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# Asymptotically log Fano varieties

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**Abstract**

Motivated by the study of Fano type varieties we define a new class of log pairs that we call asymptotically log Fano varieties and strongly asymptotically log Fano varieties. We study their properties in dimension two under an additional assumption of log smoothness, and give a complete classification of two dimensional strongly asymptotically log smooth log Fano varieties. Based on this classification we formulate an asymptotic logarithmic version of Calabi's conjecture for del Pezzo surfaces for the existence of Kähler–Einstein edge metrics in this regime. We make some initial progress towards its proof by demonstrating some existence and non-existence results, among them a generalization of Matsushima's result on the reductivity of the automorphism group of the pair, and results on log canonical thresholds of pairs. One by-product of this study is a new conjectural picture for the small angle regime and limit which reveals a rich structure in the asymptotic regime, of which a folklore conjecture concerning the case of a Fano manifold with an anticanonical divisor is a special case.

## Contents

# 1 Introduction

This article draws its motivation from classification theory of Fano type varieties in algebraic geometry on the one hand, and the uniformization problem of Kähler edge manifolds in complex differential geometry on the other hand. Our results here contribute to both of these problems, and also draw some connections between the two. In addition, we relate both of these to the theory of non-compact Calabi–Yau fibrations and the Minimal Model Program.

## 1.1 Asymptotically log Fano varieties

A projective variety $X$ is said to be *of Fano type* if there exists an effective $\mathbb{Q}$-divisor

$$\Delta = \sum_{i=1}^{r} a_i \Delta_i$$

on $X$ such that the divisor $-K_X - \Delta$ is ample and the pair $(X, \Delta)$ has at most Kawamata log terminal singularities [39, Lemma–Definition 2.6]. Fano type varieties possess very nice properties: they are rationally connected [47], they are Mori dream spaces [2], and their Cox rings have mild singularities [3], [17]. Moreover, Fano type varieties play an important role in birational geometry: they are building blocks of rationally connected varieties [30], they appears as exceptional divisors of extremal contractions and they behave well under contractions [39, Lemma 2.8].

Can we classify Fano type varieties? Probably not. This problem seems to be beyond current reach even in dimension two. One can expect that the problem is much easier if we restrict ourself to the *log smooth* case, i.e., when $X$ is smooth and the support of $\Delta$ is a simple normal crossing divisor. However, this does not seem to be the case, and the later problem seems equally hard and is also very far from being solved even in dimension two.

One of the early attempts at classifying pairs with such properties is Maeda's work. Maeda coined the term "log Fano varieties" for log smooth pairs $(X, D)$ such that $-K_X - D$ is ample and gave a complete classification in dimensions two and three [33]. A special family of two-dimensional Fano type varieties whose boundaries have *standard* coefficients, i.e., all $a_i$ are of the form $\frac{m-1}{m}$ for $m \in \mathbb{N}$, appeared naturally in the work of Kollár who used them to construct 5-dimensional real manifolds that carry an Einstein metric with positive constant [28, 29]. In a different setting, work of Tsuji, Tian, and Donaldson, suggests to consider pairs $(X, D)$ where $X$ is itself Fano, and $D$ is an anticanonical divisor whose boundaries have real coefficients close to 1 [46, 44, 14]. Then the numbers $2\pi(1-a_i)$ have a natural concrete geometrical interpretation by considering Kähler metrics with positive curvature that have edge singularities along $\Delta$, in other words metrics modelled on a one-dimensional cone of angle $2\pi(1-a_i)$ along each 'complex edge' $\Delta_i$. Such metrics were introduced by Tian as a natural generalization of conical Riemann surfaces. A general existence theorem for Kähler–Einstein edge (KEE) metrics with a smooth divisor has been obtained by Jeffres–Mazzeo–Rubinstein [24] and we come back to this circle of ideas in §1.3–1.4.

The present work draws its motivation from all three of these geometric settings: the asymptotic classes we introduce next contain as special cases these previously studied geometries.

**Definition 1.1.** *We say that a pair $(X, D)$ consisting of a projective variety $X$ with $-K_X$ $\mathbb{Q}$-Cartier and a divisor $D = \sum_{i=1}^{r} D_i$ (where the $D_i$ are distinct $\mathbb{Q}$-Cartier prime Weyl divisors) on $X$ is (strongly) asymptotically log Fano if the log pair $(X, (1-\beta_i)D_i)$ has Kawamata log terminal singularities, and the divisor $-K_X - \sum_{i=1}^{r}(1-\beta_i)D_i$ is ample for (all) sufficiently small $(\beta_1, \ldots, \beta_r) \in (0, 1]^r$.*

In the two dimensional case, we also refer to such pairs as *(strongly) asymptotically log del Pezzo.* Note that both definitions (asymptotically log Fano and strongly asymptotically log Fano) coincide if $D$ consists of a single component. This is not the case when $D$ is reducible.

For the rest of this article we restrict without further mention to the (already challenging) log smooth case, i.e., when $X$ is smooth and $D$ has simple normal crossings.

### 1.2 Classification results in dimension two

In this article we classify all strongly asymptotically log del Pezzo surfaces, i.e., we explicitly describe all pairs $(S, C)$ consisting of a smooth surface $S$ and a simple normal crossing curve $C$ on $S$ such that $(S, C)$ is strongly asymptotically log del Pezzo. We believe many of the results and techniques presented should also be useful for classifying all asymptotically log del Pezzo surfaces in the future. Our main classification result is as follows.

**Theorem 1.2.** *Let $S$ be a smooth complex surface. Let $C = C_1 + \ldots C_r$ be a simple normal crossing divisor on $S$, with each of the $C_i$ smooth. Then $(S, C)$ is a strongly asymptotically log del Pezzo surface if and only if it is one of the pairs listed in Theorem 2.1 (when r =1) or Theorem 3.1 ($r \geq 2$).*

This generalizes the classical result of Castelnuovo, Enriques and del Pezzo for the case with no boundary [9, 21], as well as its generalization to the logarithmic setting by Maeda [33] who classified all pairs $(S, C)$ with $-K_S - C$ ample.

The classification part ('only if') of the proof occupies Sections 2 ($r = 1$) and 3 ($r \geq 2$). The first several steps are to obtain useful topological and cohomological restrictions on the boundary curve. For instance, $C$ has genus at most one, and when it is elliptic it must be anticanonical, $r$ must be 1, and $S$ must be del Pezzo (Lemmas 2.2 and 3.2). Thus, we may assume that $C \not\sim -K_S$ and that the $C_i$ are rational. Then $C^2 \leq -2$, i.e., $C$ 'traps' the negative curvature portion of $-K_S$ (Lemma 2.6). In the same token, no other rational curve may have self-intersection less than $-1$ (Lemma 2.5), reflecting the fact that the curvature should morally be positive outside of $C$. But $-1$-curves are indeed allowed away from $C$ and an important task is to understand their geometry and configuration relative to $C$. Lemma 2.7 shows that such curves come in two types: disjoint from $C$ or intersecting it transversally at exactly one point. Motivated by this observation we say a pair is *minimal* if it contains no $-1$-curves of the second type. Lemmas 2.11 and 3.13 show that minimality implies the Picard group is 'small', namely, of rank at most 2. The case $r \geq 2$ relies on some general results (proved in §3.1) on the combinatorial and cohomological structure of the boundary that hold also in the asymptotic (and not necessarily strongly asymptotic) regime. Thus, we perform an induction on $\mathrm{rk}(\mathrm{Pic}(S))$ by successively contracting $-1$-curves; the observation that makes this possible is that when $-1$-curves of the first type are contracted the resulting pair is still log smooth and strongly asymptotically del Pezzo (Lemmas 2.10, 3.4, and 3.12). An additional complication in the case $r \geq 2$ is that the blown-down $-1$-curve could be a component of the boundary. According to Lemma 3.6 such a curve must be at the 'tail': it cannot intersect two boundary components. Then Lemma 3.13 guarantees the inductive step can still be carried out. Once this induction has been carried out all that remains is to classify all pairs with $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$ (Lemmas 2.9 and 3.11).

The second part of the proof of Theorem 1.2 consists of the verification that each pair appearing in the lists of Theorems 2.1 and 3.1 is strongly asymptotically log del Pezzo (§4.2). Instead of checking each case separately, we approach this straightforward task slightly more systematically by first reformulating those two lists in a unified list (Theorem 1.4) according to the positivity of the logarithmic anticanonical bundle $-K_S - C$—this is discussed in detail in the next paragraph. When this bundle is trivial or ample the verification is then immediate. In the remaining two cases (big but not ample, and nef but not big) we verify case by case.

The classification theorem has a number of corollaries, but we state here only the most obvious one.

**Corollary 1.3.** *Let $(S, C)$ be a log smooth strongly asymptotically log del Pezzo pair. Then $C$ contains at most four components.*

It would be interesting to find a similar bound in all dimensions. In the simpler log Fano setting of Maeda, a pair $(X, D)$ induces by restriction a log Fano pair of one dimension lower, and so by induction the number of components is bounded by $\dim X$ [33, Lemma 2.4].

The classification of strongly asymptotically log del Pezzo surfaces according to the positivity of the logarithmic anticanonical bundle just mentioned plays a crucial role also in other parts of this article and so we now state it precisely. We distinguish between four mutually exclusive classes. Class ($\aleph$): $S$ is del Pezzo and $C \sim -K_S$; class ($\beth$): $C \not\sim -K_S$ and $(K_S + C)^2 = 0$; class ($\gimel$): $-K_S - C$ is big but not ample; class ($\daleth$): $-K_S - C$ is ample.

**Theorem 1.4.** *Strongly asymptotically log del Pezzo pairs, whose list appears in Theorems 2.1 and 3.1, are classified according to the positivity properties* ($\aleph$), ($\beth$), ($\gimel$), *and* ($\daleth$) *as follows:*

($\aleph$) $(S, \sum_{i=1}^r C_i)$ *is one of* (I.1A), (I.4A), (I.5.m), (II.1A), (II.4), (II.5A.m), (II.8.m), (III.1), (III.2), (III.4.m) *or* (IV),

($\beth$) $(S, \sum_{i=1}^r C_i)$ *is one of* (I.3A), (I.4B), (I.9B.m), (II.2A.n), (II.2B.n), (II.3), (II.6A.n.m), (II.6B.n.m), (II.7.m), (III.3.n) *or* (III.5.n.m),

(ג) $(S, \sum_{i=1}^r C_i)$ *is one of* (I.6B.m), (I.6C.m), (I.7.n.m), (I.8B.m), (I.9C.m), (II.5B.m) *or* (II.6C.n.m),

(ד) $(S, \sum_{i=1}^r C_i)$ *is one of* (I.1B), (I.1C), (I.3B), (I.2.n), (I.4C), (II.1B) *or* (II.2C.n).

The verification of this list is an elementary corollary of Theorems 2.1 and 3.1 and appears in §4.1. It can be seen as a generalization of two previously known classes. Class (ד) is Maeda's classical classification of what he coined as 'log del Pezzo surfaces' [33]. On the other hand, the class (א) is simply the classical class of del Pezzo surfaces together with the information of a simple normal crossing anticanonical curve but its explicit (and very elementary) classification seems to appear here for the first time. The classes (ב) and (ג) are new.

## 1.3 An asymptotic logarithmic version of Calabi's conjecture

In 1990, in what became known as the resolution of Calabi's conjecture for del Pezzo surfaces, Tian gave a complete classification of those complex surfaces that admit a smooth KE metric of positive curvature [43]. In light of Theorem 1.2 it is therefore very natural and tempting to hope for a counterpart for strongly asymptotically log del Pezzo surfaces. One of the main goals of this article is to formulate such a conjecture as well as prove key parts of it. As might be expected, the situation in the singular setting is quite a bit more complex and we intend to pursue other aspects of this conjecture in future work.

As we now explain in detail, a surprisingly accurate guide to this uniformization problem is the positivity classification of Theorem 1.4.

Pairs of class (א) are the best understood, since according to a result of Berman the Tian invariant of the pair is then bigger than $\frac{n}{n+1}$, which subsequently implies by the work of Jeffres–Mazzeo–Rubinstein (Theorem 1.14 below, cf. [36, Corollary 1.5]) that the pair admits KEE metrics for all small angles. We generalize Berman's result in several ways by obtaining a general bound on the global log canonical threshold in a possibly singular and/or degenerate setting (Proposition 6.5). This gives an algebraic proof of the aforementioned estimate due to Berman for the class (א) with explicit (but far from optimal) lower bounds on the largest angle possible in dimensions two and three (Proposition 6.10).

The uniformization problem is thus reduced to understanding the existence problem for pairs of classes (ב), (ג), and (ד).

As a first guide, we investigate the asymptotic behavior in the small-angle limit of Tian's invariant $\alpha(X, (1-\beta)D)$, also refereed to as the global log canonical threshold of the pair $(X, D)$ (see §6.1 for definitions).

**Theorem 1.5.** *Assume $(S, C)$ is asymptotically log del Pezzo with $C$ smooth and irreducible. Then*

$$\lim_{\beta \to 0^+} \alpha(S, (1-\beta)C) = \begin{cases} 1 & \textit{class } (\aleph), \\ 1/2 & \textit{class } (\beth), \\ 0 & \textit{class } (\gimel) \textit{ or } (\daleth) \end{cases}$$

This gives an indication that the existence theory might, in fact, depend on the positivity classification. In fact, we conjecture that the positivity classification completely determines the existence problem.

**Conjecture 1.6.** *Suppose that $(S, C)$ is strongly asymptotically log del Pezzo with $C$ smooth and irreducible. Then $S$ admits Kähler–Einstein edge metrics with angle $\beta$ along $C$ for all sufficiently small $\beta$ if and only if $(K_S + C)^2 = 0$, i.e., $(S, C)$ is of class (א) or (ב).*

To put this conjecture in appropriate context and give perhaps more striking motivation for its validity we begin by noting that $0, 1/2$ and $1$ are the Tian invariants of $\mathbb{P}^n, n \to \infty, \mathbb{P}^1$, and $\mathbb{P}^0$, respectively. It is then tempting to think of $1/2$ as the Tian invariant of certain generic rational fiber. Motivated by this we prove the following structure theorem for surfaces of class ($\beth$).

**Proposition 1.7.** *If $(K_S + \sum_{i=1}^r C_i)^2 = 0$, then the linear system $|-(K_S + \sum_{i=1}^r C_i)|$ is free from base points and gives a morphism $S \to \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$, and every reducible fiber consists of exactly two components, each a $\mathbb{P}^1$.*

Thus, surfaces of class ($\beth$) are conic bundles, and the boundary $C$ intersects each generic fiber at two points. This gives strong motivation for the 'if' part of Conjecture 1.6 because it suggests what the small-angle limit of the purported KEE metrics on pairs of class ($\beth$) could be:

**Conjecture 1.8.** *Let $(S, C, \omega_\beta)$ be KEE pairs of class ($\aleph$) or ($\beth$). Then $(S, C, \omega_\beta)$ converges in an appropriate sense to a a generalized KE metric $\omega_\infty$ on $S \setminus C$ as $\beta$ tends to zero. In particular, $\omega_\infty$ is a Calabi–Yau metric in case ($\aleph$), and a cylinder along each generic fiber in case ($\beth$).*

The generalized KE metrics alluded to in the conjecture are related to metrics studied by Song–Tian on elliptic fibrations [41], however there are some important differences. We postpone an in-depth discussion of this to a sequel.

This conjecture can be generalized to any dimension, and is perhaps better understood in such a more general context. To that end we first note that Proposition 1.7 is a very special and explicit case of a much more general result that is a direct corollary of deep results of Kawamata and Shokurov.

**Theorem 1.9.** *Suppose $(X, D)$ is asymptotically log Fano and $-K_X - D$ is not big. Then $|-n(K_X + D)|$ is base point free for $n \gg 1$ and gives a morphism $\phi\colon X \to Z$ whose general fiber $F$ is a Fano type variety. Moreover, $D|_F \sim_{\mathbb{Q}} -K_F$ and $(F, D|_F)$ is asymptotically log Fano. Furthermore, if $(X, D)$ is strongly asymptotically log Fano, then $F$ is a Fano variety with Kawamata log terminal singularities, and $(F, D|_F)$ is strongly asymptotically log Fano.*

*Proof.* By Kawamata–Shokurov's Basepoint-free Theorem [30, Theorem 3.3] the linear system $|-n(K_X+D)|$ is base point free for $n \gg 1$. Let $\phi\colon X \to Z$ be a morphism given by it, and let $F$ be its general fiber. If $(X, \sum_{i=1}^r (1-\beta_i)D_i)$ is Kawamata log terminal and $-K_X - \sum_{i=1}^r (1-\beta_i)D_i$ is ample, then $(F, \sum_{i=1}^r (1-\beta_i)D_i|_F)$ has at most Kawamata log terminal singularities and

$$-\Big(K_F + \sum_{i=1}^r (1-\beta_i)D_i|_F\Big) \sim_{\mathbb{R}} -\Big(K_X + \sum_{i=1}^r (1-\beta_i)D\Big)\Big|_F$$

is ample. Thus, $(F, D|_F)$ is asymptotically log Fano. Note that by using adjunction $D|_F \sim_{\mathbb{Q}} -K_F$, because $F$ is a fiber of $\phi$ and $\phi$ is given by $|-n(K_X + D)|$.

If $(X, D)$ is strongly asymptotically log Fano the same argument shows that so is $(F, D|_F)$. Moreover, then

$$-K_F \sim -K_X|_F \sim_{\mathbb{Q}} D|_F \sim_{\mathbb{R}} \frac{1}{\beta}\beta D|_F \sim_{\mathbb{R}} \frac{1}{\beta}(-K_X - D - \beta D)|_F = -(K_X + \sum_{i=1}^r (1-\beta)D)|_F$$

for small $\beta \in (0, 1]$, which implies that $-K_F$ is ample, i.e., $F$ is a Fano variety. $\square$

**Corollary 1.10.** *Let $X$ be a smooth variety, let $D$ be smooth and irreducible Weil divisor on $X$. Suppose $(X, D)$ is asymptotically log Fano and $-K_X - D$ is not big. Then $|-n(K_X + D)|$ is base point free for $n \gg 1$ and gives a morphism $\phi\colon X \to Z$ whose general fiber $F$ is a smooth Fano variety with $D|_F \in |-K_F|$.*

Therefore, we conjecture:

**Conjecture 1.11.** *Suppose that $(X, D)$ is strongly asymptotically log Fano manifold with $D$ smooth and irreducible. Let $\kappa := \inf\{\mathbb{N} \ni k \le \dim X \,:\, (K_X + D)^k = 0\}$.*
*(i) There exist no KEE metric with small $\beta$ if $\kappa = \infty$.*
*(ii) Suppose that $(K_X + D)^{\dim X} = 0$. Then there exist KEE metrics $\omega_\beta, \beta \in (0, \epsilon)$ on $(X, D)$ for some $\epsilon > 0$.*
*(iii) As $\beta$ tends to zero $(X, D, \omega_\beta)$ converges in an appropriate sense to a generalized KE metric $\omega_\infty$ on $X \setminus D$ that is Calabi–Yau along its generic $(\dim X + 1 - \kappa)$-dimensional fibers.*
*(iv) Furthermore,*

$$\lim_{\beta\to 0^+} \alpha(X, (1-\beta)D) = \begin{cases} 1 & \text{if } K_X + D \sim 0, \\ \min\{1, \alpha(X, [-K_X - D]), \alpha(D)\} & \text{if } 0 \not\sim -K_X - D \text{ is not big}, \\ 0 & \text{if } -K_X - D \text{ is big}. \end{cases} \tag{1.1}$$

Conjecture 1.11 (iii) is itself a generalization of a folklore conjecture in Kähler geometry for the case $\kappa = 1$ mentioned, e.g., by Donaldson [14, p. 76], saying that $X \setminus D$ equipped with the Tian–Yau metric [45] should be a limit of KEE metrics on $(X, D)$ when $X$ is Fano and $D \in |-K_X|$. As mentioned earlier, Conjecture 1.11 (ii) holds when $\kappa = 1$. In Propostion 6.10 we further give explicit bounds on $\epsilon$ when $\dim X \in \{2, 3\}$ and $\kappa = 1$.

Since the work of Hitchin, Kobayashi, and many others, a standard condition for the existence of canonical metrics that can be described as zeros of an infinite-dimensional moment map is some sort of 'stability' condition. How, then, does Conjecture 1.11 fit into this scheme? The condition $(K_X + D)^{\dim X} = 0$ hardly looks at first like a stability condition. Perhaps one way to motivate it is to conceive the non-compact Calabi–Yau fibration of Conjecture 1.11 (ii) as a KEE metric itself, only with $\beta = 0$. In case (i) such a smooth (and hence non-compact) limit does not exist since too much 'positivity' is still remaining, and so the small angle regime, which would otherwise be a metric 'perturbation' of that limit, should not exist either. Thus, the existence of the Calabi–Yau degeneration provides the necessary 'stability' in this situation, at least conjecturally. An obvious advantage of the existence criterion of Conjecture 1.11 is that it is very explicit as opposed to logarithmic K-stability which in general seems hard to check.

We prove Conjecture 1.11 (iv) except for the middle case which we only prove in dimension two (Propositions 6.8, 6.9, and 6.10). We refer to §6.2 for one technical issue relevant to the definition of the invariants appearing in (1.1).

Finally, we make some progress towards Conjecture 1.11 (i) and (ii) in dimension two, i.e., Conjecture 1.6, that we describe next.

### 1.4 Existence and non-existence results in the asymptotic regime

Matsushima's theorem [34] implies that the Kähler–Einstein metric is the most aesthetically pleasing one since it exhibits the maximal symmetry possible: every one-parameter subgroup of automorphisms of the complex structure $\mathrm{Aut}(X)$ can be realized as the complexification of a one-parameter subgroup of isometries of the KE metric. This has a natural generalization to the edge setting by considering the automorphism group of the pair $\mathrm{Aut}(X, D)$, i.e., elements of $\mathrm{Aut}(X)$ that map $D$ to itself.

**Theorem 1.12.** *Let $(X, D, g)$ be a KEE manifold. Then $\mathrm{Aut}_0(X, D) = \mathrm{Isom}_0(X, g)^{\mathbb{C}}$. In particular, $\mathrm{Aut}_0(X, D)$ is reductive.*

Here $\mathrm{Isom}_0(X, g)^{\mathbb{C}}$ denotes the complexification of the identity component of the isometry group of $(X, g)$, while $\mathrm{Aut}_0(X, D)$ denotes the identity component of $\mathrm{Aut}(X, D)$. Theorem 1.12 is proved in Section 5, using the asymptotic structure of solutions to linear elliptic equations with edge degeneracies in the sense of Mazzeo [35] as developed in the complex codimension one setting in [24].

In the smooth world, Matsushima's criterion is often considered as a rather coarse obstruction to existence. Nevertheless, in the asymptotic regime with its much richer variety of cases, such a tool proves to be quite useful.

**Theorem 1.13.** *The following strongly asymptotically log del Pezzo pairs listed in Theorem 2.1 do not admit KEE metrics for sufficiently small $\beta$:* (I.1C)*,* (I.2.n) *with any $n \geq 0$,* (I.6C.m) *with any $m \geq 1$,* (I.7.n.m) *with any $n \geq 0$ and $m \geq 1$,* (I.6B.1)*,* (I.8B.1) *and* (I.9C.1)*.*

This proves part of the 'only if' direction of Conjecture 1.6. It is proven in Section 7.1 by computing the automorphisms groups of pairs of classes ($\gimel$) and ($\daleth$). We also supply further evidence for the converse direction of the conjecture by showing that all pairs of class ($\beth$) have reductive automorphism groups (Theorem 7.2).

Next, we turn to the existence part of Conjecture 1.6. Our main tool here is the following existence theorem that is a special case of [24, Theorem 2, Lemma 6.13]. The invariant $\alpha_G(S, (1-\beta)C)$ is the $G$-invariant Tian invariant of the pair $(S, (1-\beta)C)$ with respect to the Kähler class $[-K_S - (1-\beta)C]$ (see Definition 7.3).

**Theorem 1.14.** *Let $(S, C)$ be a strongly asymptotically log del Pezzo surface with $C$ smooth and irreducible. Suppose that $G \subset \mathrm{Aut}(S)$ is a finite group and that $\alpha_G(S, (1-\beta)C) > 2/3$. Then there exists a Kähler–Einstein edge metric with positive Ricci curvature and with angle $2\pi\beta$ along $C$.*

We apply this to prove the following existence theorem for pairs of class ($\beth$) giving the first construction of KEE metrics of positive curvature and of small angle outside of the classical class ($\aleph$).

**Theorem 1.15.** *There exist strongly asymptotically log del Pezzo pairs of type (I.3A), (I.4B), and (I.9B.5) (listed in Theorem 2.1) that admit KEE metrics for all sufficiently small $\beta$.*

This result is proven using computations of the Tian invariant of these pairs (Subsection 7.2). In the cases (I.3A), (I.4B) the pair possesses certain discrete symmetry that allows using representation theoretic arguments coupled with Shokurov's connectedness principle for log canonical loci to conclude that in fact Tian's invariant equals 1 for all $\beta \in (0, 1]$. The case (I.9B.5) is somewhat more delicate since then $S$ varies in a moduli space. We choose the Clebsch cubic surface in that space and again are able to show that the Tian invariant equals 1 for all $\beta \in (0, 1]$. We also compute the Tian invariant of more general cubic surfaces with an Eckardt point and show that without symmetry one cannot apply the existence result of Theorem 1.14. This last computation (Proposition 7.6) generalizes a result from the smooth setting [5, Theorem 1.7].

Using log slope stability Li–Sun proved that the pairs (I.1B) and (I.3B) admit no KEE metrics for small $\beta$ [32, §3]. It is possible to apply arguments similar to theirs to prove non-existence results for other pairs of class ($\gimel$) and ($\daleth$) but for the sake of brevity we postpone this discussion, along with further existence results for class ($\beth$), to a separate article.

### 1.5 Conventions

Let us describe notation and basic results that will be used throughout the article.

By a curve in an algebraic variety $X$ we mean an irreducible reduced subvariety of dimension one. Occasionally, we allow curves to be reducible (but we always assume that they are reduced). For a curve $C$ on a smooth surface $S$, we define its arithmetic genus $p_a(C)$ by

$$p_a = h^1(O_C). \tag{1.2}$$

Then $2p_a(C) - 2 = K_S \cdot C + C^2$ by [18]. When $C$ is smooth $p_a(C)$ equals the genus of $C$, $g(C)$. If $C$ is an irreducible curve on a smooth surface $S$, then by applying adjunction one verifies that

$$C \cong \mathbb{P}^1 \text{ if and only if } p_a(C) = 0 \tag{1.3}$$

This can be quite handy.

By $\sim$ we assume rational equivalence of Weil divisors or Cartier divisors (or their classes in $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$, respectively) except in Section 5 where $\sim$ stands for equality in the sense of complete asymptotic expansions as in [24]. By $\sim_{\mathbb{Q}}$ we assume $\mathbb{Q}$-rational equivalence of $\mathbb{Q}$-divisors, i.e., $D_1 \sim_{\mathbb{Q}} D_2$ if and only if $nD_1 \sim nD_2$ for some non-zero integer $n$ such that $nD_1$ and $nD_2$ are integral divisors. By $\mathbb{Q}$-Cartier and $\mathbb{R}$-Cartier divisors we mean elements in $\mathrm{Pic}(X) \otimes \mathbb{Q}$ and $\mathrm{Pic}(X) \otimes \mathbb{R}$, respectively. By $\sim_{\mathbb{R}}$ we assume $\mathbb{Q}$-rational equivalence of $\mathbb{R}$-divisors, i.e., $D_1 - D_2$ is a sum with real coefficients of $\mathbb{Q}$-Cartier divisors that are $\mathbb{Q}$-rationally equivalent to zero.

For two divisors $D_1$ and $D_2$, we write $D_1 \equiv D_2$ (and say that $D_1$ and $D_2$ are numerically equivalent) iff $D_1 - D_2$ is $\mathbb{R}$-Cartier divisor such that $(D_1 - D_2).C = 0$ for every curve $C \subset X$. Vice versa, we say that two curves $C_1$ and $C_2$ on $X$ are numerically equivalent iff $D.C_1 = D.C_2$ for every $\mathbb{R}$-Cartier divisor $D$ on $X$. Similarly, we define numerical equivalence of 1-cycles (with real or rational coefficients) on $X$. We denote the real vector space of 1-cycles modulo numerical equivalence by $\mathrm{N}_1(X)$. By the cone of curves or the Mori cone of $X$ we assume the cone in $\mathrm{N}_1(X)$ generated by curves in $X$. We denote the Mori cone of $X$ by $\mathrm{NE}(X)$. By $\overline{\mathrm{NE}(X)}$ we denote its closure.

Recall that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is called ample if there exists positive integer $n$ such that $nD$ is a very ample Cartier divisor. By Kleiman's criterion, $D$ is ample if and only if $D$ is positive on $\overline{\mathrm{NE}(X)}$ (and this in turn is equivalent to the differential geometric notion of positivity of a class). The latter can be used as a definition of ampleness for $\mathbb{R}$–Cartier $\mathbb{R}$-divisors. Note that in the case of surfaces, the ampleness of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is equivalent to

$$D^2 > 0 \text{ and } D.C > 0 \tag{1.4}$$

for every curve $C \subset X$. So we can use the latter condition as another definition of ampleness for $\mathbb{R}$–Cartier $\mathbb{R}$-divisors on surfaces. This criterion-definition is very handy for surfaces: if $D$ is an ample $\mathbb{R}$-Cartier divisor on a smooth surface $S$, then

$$\pi_\star(D) \text{ is an ample } \mathbb{R}\text{-Cartier divisor} \tag{1.5}$$

for every birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface.

Recall that a $\mathbb{Q}$-divisor $D$ is called big if $h^0(\mathcal{O}_X(nD))$ grows as $O(n^{\dim(X)})$ for $n \gg 1$ such that $nD$ is an integral divisor. One can show that $D$ is big if and only if it is a sum of an effective divisor and an ample divisor. For $\mathbb{R}$-divisors this can be used as a definition of bigness.

Recall that a divisor $D$ is effective if $D$ is a finite linear combination of prime Weil divisors with non-negative coefficients, and that $h^0(\mathcal{O}_X(-D)) = 0$ for every non-zero effective Weil divisor $D$. An $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D$ is called nef (a shortcut for numerically effective) if $D.C \geq 0$ for every curve $C \subset X$. Thus,

$$\text{if } D \text{ is effective and } -D \text{ is nef, then } D \text{ is a zero divisor.} \tag{1.6}$$

For each $n \geq 0$, denote by

$$\mathbb{F}_n \tag{1.7}$$

the unique rational ruled surface whose Picard group has rank two and contains a unique (if $n > 0$) smooth rational curve of self-intersection $-n$. We denote this curve by $Z_n$, and by $F$ we denote an irreducible smooth rational curve such that $F^2 = 0$ and $F.Z_n = 1$. If $n = 0$ when we refer to $Z_0$ and $F$ we intend that each is a fiber of a different projection to $\mathbb{P}^1$. Such a surface can be constructed, e.g., as a toric variety or as a ruled surface and [20, Chapter 5, §2] and applying adjunction yields

$$-K_S \sim 2Z_n + (n+2)F \tag{1.8}$$

Recall that every smooth irreducible curve in $|Z_n + nF|$ (a 'zero section') intersects each fiber transversally at a single point and does not intersect the 'infinity section' $Z_n$. Any curve $C$ on $\mathbb{F}_n$ satisfies $C \sim aZ_n + bF$ with $a, b \in \mathbb{N} \cup \{0\}$. This, combined with (1.4), implies

$$C \text{ is ample if and only if } a > 0 \text{ and } b > na, \tag{1.9}$$

and furthermore,

$$C \text{ is an irreducible curve if and only if } C = Z_n \text{ or } b \geq na \geq 0. \tag{1.10}$$

The classification of rational surfaces [18, p. 520] implies that

$$\text{every rational surface with } \mathrm{rk}(\mathrm{Pic}) > 2 \text{ contains a } -1\text{-curve,} \tag{1.11}$$

and that

$$\text{a rational surface with } \mathrm{rk}(\mathrm{Pic}) \leq 2 \text{ is either } \mathbb{P}^2 \text{ or } \mathbb{F}_n,\ n \geq 0. \tag{1.12}$$

We denote by $\mathbb{G}_a$ the additive group $(\mathbb{C}, +)$, by $\mathbb{G}_m$ the multiplicative group $(\mathbb{C}^\star, \cdot)$, and by $\mu_n$ the finite group of order $n$.

Finally, if $G$ is a graph with vertex set $V$ and edges $E$, the dual graph of $G$ refers to the graph whose vertex set is $E$ and whose edge set is $V$, namely if $v \in V, e_1, e_2 \in E$ and $v \in e_1 \cap e_2$ then $e_1$ and $e_2$ are connected in the dual graph by $v$. A graph is a cycle if for some $\mathbb{N} \ni k \geq 2$ $E = \{e_1, \ldots, e_k\}, V = \{v_1, \ldots, v_k\}$, and $e_i \cap e_{i+1} = v_i$ with $e_{k+1} := e_1$. A tree is a graph that contains no cycles. A chain is a connected tree with no three edges intersecting.

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## 2 Asymptotically log del Pezzo surfaces with smooth connected boundary

The following theorem gives complete classification in the case of a single boundary component.

**Theorem 2.1.** *Let $S$ be a smooth surface (the surface), and let $C$ be an irreducible smooth curve on $S$ (the boundary curve). Then $-K_S-(1-\beta)C$ is ample for all sufficiently small $\beta>0$ if and only if $S$ and $C$ can be described as follows:*

(I.1A) *$S\cong\mathbb{P}^2$, and $C$ is a smooth cubic elliptic curve,*

(I.1B) *$S\cong\mathbb{P}^2$, and $C$ is a smooth conic,*

(I.1C) *$S\cong\mathbb{P}^2$, and $C$ is a line,*

(I.2.n) *$S\cong\mathbb{F}_n$ for any $n\geq 0$, and $C=Z_n$,*

(I.3A) *$S\cong\mathbb{F}_1$, and $C\in|2(Z_1+F)|$,*

(I.3B) *$S\cong\mathbb{F}_1$, and $C\in|Z_1+F|$,*

(I.4A) *$S\cong\mathbb{P}^1\times\mathbb{P}^1$, and $C$ is a smooth elliptic curve of bi-degree $(2,2)$,*

(I.4B) *$S\cong\mathbb{P}^1\times\mathbb{P}^1$, and $C$ is a smooth rational curve of bi-degree $(2,1)$,*

(I.4C) *$S\cong\mathbb{P}^1\times\mathbb{P}^1$, and $C$ is a smooth rational curve of bi-degree $(1,1)$,*

(I.5.m) *$S$ is a blow-up of the surface in* (I.1A) *at $m\leq 8$ distinct points on the boundary curve such that $-K_S$ is ample, i.e., $S$ is a del Pezzo surface, and $C$ is the proper transform of the boundary curve in* (I.1A)*, i.e., $C\in|-K_S|$,*

(I.6B.m) *$S$ is a blow-up of the surface in* (I.1B) *at $m\geq 1$ distinct points on the boundary curve, and $C$ is the proper transform of the boundary curve in* (I.1B)*,*

(I.6C.m) *$S$ is a blow-up of the surface in* (I.1C) *at $m\geq 1$ distinct points on the boundary curve, and $C$ is the proper transform of the boundary curve in* (I.1C)*,*

(I.7.n.m) *$S$ is a blow-up of the surface in* (I.2.n) *at $m\geq 1$ distinct points on the boundary curve, and $C$ is the proper transform of the boundary curve in* (I.2)*,*

(I.8B.m) *$S$ is a blow-up of the surface in* (I.3B) *at $m\geq 1$ distinct points on the boundary curve, and $C$ is the proper transform of the boundary curve in* (I.3B)*,*

(I.9B.m) *$S$ is a blow-up of the surface in* (I.4B) *at $m\geq 1$ distinct points on the boundary curve with no two of them on a single curve of bi-degree* $(0,1)$*, and $C$ is the proper transform of the boundary curve in* (I.4B)*,*

(I.9C.m) *$S$ is a blow-up of the surface in* (I.4C) *at $m\geq 1$ distinct points on the boundary curve, and $C$ is the proper transform of the boundary curve in* (I.4C)*.*

The rest of the section is devoted to the proof of this theorem.

### 2.1 Classification

Throughout this subsection we assume without further mention that

$$-K_S - (1-\beta)C \text{ is ample for sufficiently small } \beta \in (0,1], \tag{2.1}$$

i.e., $(S,C)$ is asymptotically log del Pezzo. Then $-K_S - C$ is nef. Moreover, the surface $S$ is projective, since $-K_S-(1-\beta)C$ is an ample $\mathbb{Q}$-divisor for sufficiently small rational $\beta \in (0,1]$. Furthermore, the divisor $-K_S$ is big, since it is a sum of an ample class and an effective class, to wit,

$$-K_S = -(K_S + (1-\beta)C) + (1-\beta)C.$$

Since $-K_S$ is big, we have $h^0(\mathcal{O}_S(K_S)) = h^0(\mathcal{O}_S(2K_S)) = 0$. Moreover, it follows from the Kawamata–Viehweg Vanishing Theorem [31, Vol. II, §9.1.C] that $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$. Thus, the surface $S$ is rational by Castelnuovo's rationality criterion [18, p. 536]. We remark that all of these considerations apply equally when $C$ has several components, but for the rest of this subsection we implicitly assume $r=1$ unless explicitly stated.

In the rest of this subsection we prove that $(S,C)$ is one of the pairs listed in Theorem 2.1. Our proof is divided into several steps, each contained in a separate paragraph.

#### 2.1.1 Non-rational boundary

Let $g(C)$ denote the genus of the (smooth) curve $C$.

**Lemma 2.2.** *Suppose that $g(C) \neq 0$. Then $-K_S$ is ample, i.e., $S$ is a del Pezzo surface, and $C$ is a smooth elliptic curve in $|-K_S|$.*

*Proof.* Since $-K_S - C$ is nef, it follows from the adjunction theorem that

$$0 \le 2g-2 = (K_S + C).C \le 0, \tag{2.2}$$

thus $g=1$.

Next, by Kodaira–Serre duality, $h^2(\mathcal{O}_S(K_S+C)) = h^0(\mathcal{O}_S(-C)) = 0$. Also, since $S$ is rational $\chi(\mathcal{O}_S) = 1$. Recalling that $(K_S+C).C = 0$ by (2.2), and using the Riemann-Roch Theorem thus gives

$$\begin{aligned} 1 &= \chi(\mathcal{O}_S) + \frac{(K_S+C).(K_S+C-K_S)}{2} = \chi(\mathcal{O}_S(K_S+C)) \\ &= h^0(\mathcal{O}_S(K_S+C)) - h^1(\mathcal{O}_S(K_S+C)) + h^2(\mathcal{O}_S(K_S+C)) \\ &\le h^0(\mathcal{O}_S(K_S+C)). \end{aligned} \tag{2.3}$$

Therefore, there exists an effective divisor $R$ such that $R \sim K_S + C$. Thus by (1.6) $R=0$, from which $C \in |-K_S|$, and

$$-\beta K_S \sim -K_S - (1-\beta)C > 0, \tag{2.4}$$

for sufficiently small rational $\beta \in (0,1]$, i.e., $S$ is del Pezzo. $\square$

Recall that a smooth projective surface $S$ not equal to $\mathbb{P}^1 \times \mathbb{P}^1$ is del Pezzo precisely when there is a smooth anticanonical curve $C \subset S$ which is the proper transform of a smooth cubic curve in $\mathbb{P}^2$ blown-up at 8 points in general position on the cubic [21, Proposition 3.2]. Thus, we have:

**Corollary 2.3.** *Suppose $C$ is not rational, then $(S, C)$ is one of* (I.1A), (I.4A), *or* (I.5.m).

Thus, for the remainder of §2.1, we assume

$$C \text{ is a smooth rational curve.} \tag{2.5}$$

*Remark* 2.4. In the notation and assumption of Theorem 1.2, suppose additionally that $S$ is a del Pezzo surface. Then the divisor $-K_S - \sum_{i=1}^r (1-\beta_i)C_i$ is ample for *any* $(\beta_1, \ldots, \beta_r) \in (0,1]^r$.

### 2.1.2 Rational boundary and curves of negative self-intersection

The goal is now to show that $(S, C)$ is one of the cases not covered by Corollary 2.3 . In this paragraph we derive some basic intersection properties of the boundary and other curves of negative self-intersection.

**Lemma 2.5.** *Let $Z$ be an irreducible curve on $S$ such that $Z \neq C$ and $Z^2 < 0$. Then $Z$ is a smooth rational curve and $Z^2 = -1$.*

*Proof.* Since $Z \neq C$,

$$-K_S.Z = -\Big(K_S + (1-\beta)C\Big).Z + (1-\beta)C.Z \geq -\Big(K_S + (1-\beta)C\Big).Z > 0, \tag{2.6}$$

for sufficiently small $\beta \in (0, 1]$. Hence, by (1.2) it follows that

$$0 > Z^2 > K_S.Z + Z^2 = 2h^1(\mathcal{O}_Z) - 2, \tag{2.7}$$

or $h^1(\mathcal{O}_Z) = 0$. Now it follows from (1.3) that $Z$ is a smooth rational curve. Going back to (2.7) then $Z^2 = -1$. □

**Lemma 2.6.** *Suppose that $S$ is not a del Pezzo surface. Then $C^2 \leq -2$.*

*Proof.* Suppose that $C^2 \geq -1$. Then $-K_S.C > 0$ by the adjunction formula, since by Lemma 2.5 $C$ is a smooth rational curve. Also, by (2.6) $K_S.Z < 0$ for every irreducible curve $Z \neq C$ on the surface $S$. Moreover, $K_S^2 > 0$, since $-K_S$ is big. Therefore, the divisor $-K_S$ is ample by the Nakai–Moishezon criterion, which contradicts our assumption. □

The next lemma is crucial for the proof of the main result. It shows that any $-1$-curve intersects the boundary transversally at most at one point.

**Lemma 2.7.** *Suppose that there exists a smooth irreducible rational curve $E$ on the surface $S$ such that $E^2 = -1$ and $E \neq C$. Then either $E \cap C = \emptyset$ or $E.C = 1$.*

*Proof.* Choose $\beta$ such that $\beta C.E < 1$ (and, as always, also satisfying (2.1)). By adjunction, $-K_S.E = 1$. Then

$$0 < -(K_S + (1-\beta)C).E = 1 - C.E + \beta C.E < 2 - C.E,$$

thus $C.E < 2$. Hence, either $C.E = 0$ (and, thus $E \cap C = \emptyset$ since $E \neq C$) or $C.E = 1$. □

### 2.1.3 Minimal pairs

Suppose that there exists a smooth irreducible rational curve $E$ on the surface $S$ such that $E^2 = -1$ and $E \neq C$. By Castelnuovo's contractibility criterion there exists a birational morphism $\pi\colon S \to s$ that contracts the curve $E$ to a smooth point of the surface $s$ [18, p. 476]. Since by Lemma 2.7 $E$ and $C$ intersect transversally at most at one point then $\pi(C)$ is a smooth curve. Moreover, by (1.5), we see that the divisor $-(K_s + (1-\beta)\pi(C))$ is ample provided that $-(K_S + (1-\beta)C)$ is ample. Thus, we see that $(s, \pi(C))$ is asymptotically log del Pezzo as well.

Thus, it seems possible to use Lemma 2.7 to give an inductive proof (in the rank of the Picard group, $\mathrm{Pic}(S)$) of one direction of Theorem 2.1. To do this in a consistent way we make the following definition.

**Definition 2.8.** *The pair $(S, C)$ is* minimal *if there exist no smooth irreducible rational curve $E$ on the surface $S$ such that $E^2 = -1$, $E \neq C$ and $E \cap C \neq \emptyset$.*

The base of our induction is given by the next lemma. Recall that throughout we are assuming $C$ is a smooth rational curve (2.5).

**Lemma 2.9.** *Suppose that* $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$ *and $C \not\sim -K_S$. Then when $(S, C)$ is minimal it is one of* (I.1B)*,* (I.1C)*,* (I.2.n)*,* (I.3A)*,* (I.3B)*,* (I.4B)*, or* (I.4C)*, and otherwise it is* (I.6B.1) *or* (I.6C.1)*.*

*Proof.* First note that all the cases listed in the statement are indeed asymptotically log del Pezzo by §4.2. By (1.12) the assumption $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$ implies that either $S \cong \mathbb{P}^2$ or $S \cong \mathbb{F}_n, n \geq 0$. In the former case, $(S, C)$ is either (I.1B) or (I.1C), as $C$ is rational. Let us consider the latter cases. If $n = 0$, then one sees that $(S, C)$ is either (I.$2_0$), (I.4B), or (I.4C) (again, as $C$ is rational). Let $n > 0$ and suppose $C \in |aZ_n + bF|$ with $a, b \in \mathbb{N} \cup \{0\}$. Then by (1.8)–(1.9), $-K_S - (1-\beta)C = (2 - (1-\beta)a)Z_n + (n + 2 - (1-\beta)b)F$ is ample if and only if $a \in [0, 2], b \in [0, na + \frac{2-n}{1-\beta})$.

Suppose first that $b = 0$. Then either $(a, b) = (1, 0)$, i.e., $C = Z_n$ and we are in the case (I.2.n), or else $(a, b) = (2, 0)$, i.e., $C \in |2Z_n|$, but since $Z_n$ is unique for $n > 0$ by (1.7) this means $C$ is not reduced, so this case is excluded.

Thus it remains to consider the case $b > 0$. If $a = 0$ then necessarily $b = n = 1$. This is excluded by minimality since then $C.Z_1 = 1$ and $Z_1^2 = -1$. If $a = 1$ then $b \in [1, 2]$. The case $(a, b) = (1, 1)$ implies $C.Z_n = 1 - n \geq 0$ since $C \neq Z_n$. Thus $n = 1$ and we obtain case (I.3B). Similarly the case $(a, b) = (1, 2)$ implies $n \leq 2$. But $n = 1$ is excluded by minimality since then $C.Z_1 = 1, Z_1^2 = -1$ and $C \neq Z_1$, while $n = 2$ is excluded by Lemma 2.5 as $C \neq Z_2$. Finally, if $a = 2$ then $b \in [1, n+2]$. Then $C.Z_n = -2n + b \geq 0$ as $C \neq Z_n$. Thus either $n = 2$ and $b = 4$, or else $n = 1$ and $b = 2$ or $b = 3$. The former is again excluded by Lemma 2.5, while the latter gives only the case (I.3A) since if $(a, b) = (2, 3)$ then $C \in |-K_S|$ is not rational. $\square$

### 2.1.4 The inductive step

The next lemma provides the inductive step for our classification. Note that, by definition, part (ii) refers to the case the $-1$-curve $E$ is disjoint from the boundary.

**Lemma 2.10.** *(i) Suppose that there exists a smooth irreducible rational curve $E$ on the surface $S$ such that $E^2 = -1$ and $E \neq C$. Then there exists a birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface, $\pi(E)$ is a point, the morphism $\pi$ induces an isomorphism $S \setminus E \cong s \setminus \pi(E)$, the curve $\pi(C)$ is a smooth rational curve, and $(s, \pi(C))$ is asymptotically log del Pezzo.*
*(ii) Suppose in addition that $(S, C)$ is minimal. Then $(s, \pi(C))$ is minimal.*

*Proof.* (i) By the discussion at the beginning of §2.1.3 there exists a birational morphism $\pi\colon S \to s$ that contracts $E$ to a smooth point of the surface $s$, the curve $\pi(C)$ is a smooth rational curve, and the divisor $-(K_s + (1-\beta)\pi(C))$ is ample for sufficiently small $\beta \in (0,1]$, i.e., the pair $(s, \pi(C))$ asymptotically del Pezzo.

(ii) It remains to show that $(s, \pi(C))$ is minimal. Suppose, on the contrary, that there exists a smooth irreducible rational curve $z$ on the surface $s$ such that $z^2 = -1$, $z \neq \pi(C)$, and $z \cap \pi(C) \neq \emptyset$. Let $Z$ be the proper transform of the curve $z$ on the surface $S$. Then either $\pi(E) \in z$ and $Z^2 = -2$, contradicting Lemma 2.5, or else $\pi(E) \notin z$ and $Z^2 = -1$, but then $Z \cap C \neq \emptyset$, contradicting minimality of $(S, C)$. □

### 2.1.5 Classification of minimal pairs

The next lemma uses a geometric argument to apply the inductive step to reduce the classification of minimal pairs to Lemma 2.9.

**Lemma 2.11.** *Suppose that* $(S, C)$ *is minimal. Then* $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$.

*Proof.* If $C \sim -K_S$ then by Corollary 2.3 the pair must be (I.1A) or (I.4A), hence $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$. So we assume that $C \not\sim -K_S$.

Let $(S, C)$ be a pair and suppose that $\mathrm{rk}(\mathrm{Pic}(S)) \geq 3$. We would like to show that the pair is not minimal.

If $S$ a del Pezzo surface, there exists a smooth irreducible rational $-1$-curve $E$ on the surface $S$ such that $E \neq C$. Indeed, in this case it follows from the classification of smooth del Pezzo surfaces referred to before Corollary 2.3 that there are at least three distinct $-1$-curves in $S$ at most one of which can be the boundary. On the other hand, if $S$ is not del Pezzo, then the existence of such curve $E$ follows from Lemma 2.6, and (1.11). To complete the proof it thus suffices to show that $E \cap C \neq \emptyset$ (recall Definition 2.8).

Suppose, on the contrary, that every such $-1$-curve $E$ in $S$ satisfies $E \cap C = \emptyset$, i.e., that $(S, C)$ is minimal. By Lemma 2.10 there exists a birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface, $\pi(E)$ is a point, the morphism $\pi$ induces an isomorphism $S \setminus E \cong s \setminus \pi(E)$, the curve $\pi(C)$ is smooth, and the pair $(s, \pi(C))$ is asymptotically log del Pezzo and minimal. Since

$$\mathrm{rk}(\mathrm{Pic}(s)) = \mathrm{rk}(\mathrm{Pic}(S)) - 1 \geq 2,$$

we may as well assume that $\mathrm{rk}(\mathrm{Pic}(S)) = 3$ and $\mathrm{rk}(\mathrm{Pic}(s)) = 2$.

Since $\mathrm{rk}(\mathrm{Pic}(s)) = 2$, one has $s \cong \mathbb{F}_n$ for some $n \geq 0$. Put

$$c := \pi(C).$$

Then $\pi(E) \notin c$, since $E \cap C = \emptyset$. Since $\mathrm{rk}(\mathrm{Pic}(s)) = 2$ and $(s, c)$ is minimal, Lemma 2.9 implies that $(s, c)$ is one of (I.2.n), (I.3A), (I.3B), (I.4B), or (I.4C).

Let $\xi\colon s \to \mathbb{P}^1$ be a natural projection (unique when $n > 0$, and one of two choices when $n = 0$, see (1.7)), let $f$ be the fiber of the morphism $\xi$ that passes through the point $\pi(E)$, and let $F$ be its proper transform on $S$. Here we are following the conventions of (1.7). Then $F$ is a smooth irreducible rational curve such that $F^2 = -1$ (since $f^2 = 0$ downstairs). Moreover, we have $F \cap C = \emptyset$, since $(S, C)$ is minimal. Since $E \cap C = \emptyset$, we see that $f \cap c = \emptyset$, which implies that $c$ is also a fiber of the morphism $\xi$, i.e.,

$$c \in |f|, \tag{2.8}$$

and in particular also $c^2 = 0$. In the case (I.2.n) $c^2 = -n$, while in the cases (I.3A), (I.3B), (I.4B), (I.4C), we have $c^2 \neq 0$. Thus, $(s, c)$ is neither (I.2.n), $n > 0$, nor one of (I.3A), (I.3B), (I.4B), or (I.4C). The only remaining possibility is that $(s, c)$ is (I.2.0). Then $s \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\xi = p_1$ is a projection to one of the factors. Let $p_2$ denote the projection onto the second factor, and let $g$ denote the fiber of $p_2$ passing through $\pi(E)$. Then $g$ intersects $c$ at one point, $g^2 = 0$, and hence the proper transform of $g$, denoted $G$, satisfies $G^2 = -1$ and $G \cap C = 1$, which once again contradicts minimality of $(S, C)$. This completes the proof of the lemma. □

### 2.1.6 Dealing with non-uniqueness

Now we are ready to finish the proof of the classification part of Theorem 2.1. If $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$, then it follows from Lemma 2.9 that $(S, C)$ is one of (I.1B), (I.1C), (I.2.n), (I.3A), (I.3B), (I.4A), (I.4B), (I.4C), (I.6B.1), or (I.6C.1). Thus, we may assume that $\mathrm{rk}(\mathrm{Pic}(S)) \geq 3$. In particular, the pair $(S, C)$ is not minimal by Lemma 2.11. To prove Theorem 2.1, we must show that $(S, C)$ is one of the cases: (I.6B.m), (I.6C.m) for some $\mathbb{N} \ni m \geq 2$, (I.7.n.m) for some positive integers $n$ and $m$, or, finally, (I.8B.m), (I.9B.m), or (I.9C.m) for some positive integer $m$.

Since $(S, C)$ is not minimal, there exists a curve $E$ and a birational morphism $\pi\colon S \to s$ as in Lemma 2.10. The next lemma follows directly from Lemma 2.5.

**Lemma 2.12.** *Let $g$ be a smooth irreducible rational curve on the surface $s$ such that $g \neq \pi(C)$ and $g^2 = -1$. Then $\pi(E) \notin g$.*

Now we may replace the pair $(S, C)$ by the pair $(s, c)$ and iterate this process. As a result, we obtain a birational morphism, that by abuse, we still denote by $\pi : S \to s$ such that $s$ is a smooth surface, $\pi$ is a a blow-up of $m$ *distinct* points $P_1, P_2, \ldots, P_m$ on the smooth curve $\hat{C} \subset \hat{S}$ such that $c := \pi(C)$, and, finally, $(s, c)$ is a minimal asymptotically log del Pezzo pair. By Lemma 2.11, one has $\mathrm{rk}(\mathrm{Pic}(s)) \leq 2$. By Lemma 2.9, $(s, c)$ is (I.1B), (I.1C), (I.2.n), (I.3A), (I.3B), (I.4B), or (I.4C).

**Corollary 2.13.** *If $(s, c)$ is* (I.1B)*,* (I.1C)*,* (I.2.n)*, or* (I.4C)*, then $(s, c)$ can also be obtained as described in one of the cases* (I.6B.m)*,* (I.6C.m)*,* (I.7.n.m)*,* (I.8C.m)*, respectively.*

Thus, to complete the proof of Theorem 2.1, we must do the following two things:

- if $s \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $c$ is a smooth rational curve of bi-degree $(2, 1)$, we must check that no two points among $P_1, P_2, \ldots, P_m$ lie on a one curve in $s$ of bi-degree $(0, 1)$,
- if $(s, c)$ is (I.3A) we must show the pair $(S, C)$ can also be described by a birational morphism that is listed in Theorem 2.1.

The first point is simple. Suppose that there exist two points among $P_1, P_2, \ldots, P_m$ that lie on a one curve in $s$ of bi-degree $(0, 1)$. Let us denote this curve by $z$. Denote by $Z$ its proper transform on the surface $S$. Then $Z^2 \leq -2$, contradicting Lemma 2.5, because $z \neq c$.

The second point is dealt with using the next lemma.

**Lemma 2.14.** *Suppose that $(s, c)$ is* (I.3A)*. Then $(S, C)$ can also be described as* (I.9B.m)*.*

*Proof.* We have $s \cong \mathbb{F}_1$. Let $\xi\colon S \to \mathbb{P}^1$ be the natural projection, let $z$ be the section of $\xi$ such that $z^2 = -1$, and let $f$ be a fiber of $\xi$ that passes through the point $P_1$. Then the curve $c$ is a smooth rational in $|2z + 2f|$.

There exists a commutative diagram

$$\begin{tikzcd}
S \arrow[rr, "\upsilon"] \arrow[dr, "\pi"'] & & Bl_{P_1}\mathbb{F}_1 \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\
& s = \mathbb{F}_1 \arrow[d, "\xi"'] & & \sigma = \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d, "\xi'"] \\
& \mathbb{P}^1 \arrow[rr, equal] & & \mathbb{P}^1,
\end{tikzcd}$$

where $\psi$ is a blow-up of the point $P_1$, $\phi$ is a contraction of the proper transform of the fiber $f$, $\upsilon$ is a birational morphism, and $\xi'$ is a natural projection. Put $\pi' = \phi \circ \upsilon$. Let us show that $\pi' \colon S \to \sigma$ is the desired replacement of the birational morphism $\pi \colon S \to s$. These birational transformations did not change the generic fiber of the projection $\xi$. Thus, $\sigma$ comes equipped with a fibration $\xi' : \sigma \to \mathbb{P}^1$ In particular, the curve $\zeta$ in $\sigma$ corresponding to $z \subset s$ is a fiber of $\xi'$ and it has zero self-intersection. Thus, $\sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$. Becuase $\phi$ contracts a $-1$-curve ($\tilde{f}$, the proper transform of $f$) that intersects both $z$ and the exceptional curve $A$ of $\psi$, it follows that $\phi(A)$ has zero self-intersection and intersects $\zeta$ at one point. At the same time $\phi(A)$ intersects the transformed boundary of $s$ (which equals $\pi'(C)$ at two points. Thus $\pi'(C)$ is a curve of bi-degree $(2, 1)$. Thus, $(S, C)$ is the blow-up of $(\mathbb{P}^1 \times \mathbb{P}^1, \pi'(C))$ at $m \geq 1$ points. Further, as already checked earlier, no two of these points may lie on a single fiber of $\xi'$. Thus, $(S, C)$ is (I.9B.m). $\square$

## 3 Strongly asymptotically log del Pezzo surfaces

The following theorem gives complete classification in the case of a reducible boundary curve. We assume without further mention that in each case listed below the curves composing the boundary intersect simply and normally. A point in the smooth locus of such a boundary means a point that is not an intersection point of any two components of the boundary.

**Theorem 3.1.** *Let $S$ be a smooth surface, let $C_1, \dots, C_r$ be irreducible smooth curves on $S$ such that $\sum_{i=1}^r C_i$ is a divisor with simple normal crossings. Suppose that $r \geq 2$. Then $(S, \sum_{i=1}^r C_i)$ is a strongly asymptotically log del Pezzo surface if and only if it is one of the following pairs:*

(II.1A) $|C_1 \cap C_2| = 2$, $S \cong \mathbb{P}^2$, *and* $C_1$ *is a smooth conic, and* $C_2$ *is a line,*

(II.1B) $|C_1 \cap C_2| = 1$, $S \cong \mathbb{P}^2$, *and* $C_1$ *and* $C_2$ *are two distinct lines,*

(II.2A.n) $C_1 \cap C_2 = \emptyset$, $S \cong \mathbb{F}_n$ *for any* $n \geq 0$, $C_1 = Z_n$ *and* $C_2 \in |Z_n + nF|$,

(II.2B.n) $|C_1 \cap C_2| = 1$, $S \cong \mathbb{F}_n$ *for any* $n \geq 0$, $C_1 = Z_n$ *and* $C_2 \in |Z_n + (n+1)F|$,

(II.2C.n) $|C_1 \cap C_2| = 1$, $S \cong \mathbb{F}_n$ *for any* $n \geq 0$, $C_1 = Z_n$ *and* $C_2 = F$,

(II.3) $|C_1 \cap C_2| = 1$, $S \cong \mathbb{F}_1$, $C_1, C_2 \in |Z_1 + F|$,

(II.4A) $|C_1 \cap C_2| = 2$, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, $C_1, C_2$ *are distinct bi-degree* $(1, 1)$ *curves,*

(II.4B) $|C_1 \cap C_2| = 2$, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, *the curve* $C_1$ *is a smooth rational curve of bi-degree* $(2, 1)$, *and* $C_2$ *is a smooth rational curve of bi-degree* $(0, 1)$, (II.5A.m) $|C_1 \cap C_2| = 2$, *$(S, C)$ is a blow-up of* (II.1A) *at $1 \leq m \leq 5$ points in the smooth locus of the boundary curve such that the surface $S$ is a del Pezzo surface and $C_1^2, C_2^2 \geq 0$, i.e., $C_1 + C_2 \sim -K_S$, and there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ such that $\pi(C_1)$ is a smooth conic, and $\pi(C_2)$ is a line such that $|\pi(C_1) \cap \pi(C_2)| = 2$, and $\pi$ is a blow-up of $1 \leq m \leq 5$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ but away from $\pi(C_1) \cap \pi(C_2)$ with no two of them on $\pi(C_1)$, and no five of them on $\pi(C_2)$,*

(II.5B.m) $|C_1 \cap C_2| = 1$, *$(S, C)$ is a blow-up of* (II.1B) *at $m \geq 1$ points in the smooth locus of the boundary curve, i.e., there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ such that $\pi(C_1)$ and $\pi(C_2)$ are distinct lines, and $\pi$ is a blow-up of $m \geq 1$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ but away from $\pi(C_1) \cap \pi(C_2)$,*

(II.6A.n.m) $C_1 \cap C_2 = \emptyset$, *$(S, C)$ is a blow-up of* (II.2A.n) *at $m \geq 1$ points on the boundary curve such that there exists a birational morphism $\pi\colon S \to \mathbb{F}_n$ for some $n \geq 0$ such that $\pi(C_1) = Z_n$, $\pi(C_2) \in |Z_n + nF|$, and $\pi$ is a blow-up of $m$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ with at most one point on a single curve in the linear system $|F|$,*

(II.6B.n.m) $|C_1 \cap C_2| = 1$, *$(S, C)$ is a blow-up of* (II.2B.n) *at $m \geq 1$ points in the smooth locus of the boundary curve such that there exists a birational morphism $\pi\colon S \to \mathbb{F}_n$ for some $n \geq 0$ such that $\pi(C_1) = Z_n$, $\pi(C_2) \in |Z_n + (n+1)F|$, and $\pi$ is a blow-up of $m \geq 1$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ with at most one point on a single curve in the linear system $|F|$, and no point being $\pi(C_1) \cap \pi(C_2)$*

(II.6C.n.m) $|C_1 \cap C_2| = 1$, *$(S, C)$ is a blow-up of* (II.2C.n) *at $m \geq 1$ points in the smooth locus of the boundary curve, i.e., there exists a birational morphism $\pi\colon S \to \mathbb{F}_n$ for some $n \geq 0$ such that $\pi(C_1) = Z_n$, $\pi(C_2) = F$, and $\pi$ is a blow-up of $m \geq 1$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ but away from $\pi(C_1) \cap \pi(C_2)$,*

(II.7.m) $|C_1 \cap C_2| = 1$, *$(S, C)$ is a blow-up of* (II.3) *at $m \geq 1$ points in the smooth locus of the boundary curve such that there exists a birational morphism $\pi\colon S \to \mathbb{F}_1$ such that $\pi(C_1), \pi(C_2) \in |Z_1 + F|$, and $\pi$ is a blow-up of $m \geq 1$ distinct points on $\pi(C_1)$ and $\pi(C_2)$ with at most one point on a single curve in the linear system $|F|$, and no point being $\pi(C_1) \cap \pi(C_2)$*

(II.8.m) $|C_1 \cap C_2| = 2$, *$(S, C)$ is a blow-up of* (II.4B) *at $1 \leq m \leq 4$ points in the smooth locus of the boundary curve such that $S$ is a del Pezzo surface and $C_1^2, C_2^2 \geq 0$, i.e., $C_1 + C_2 \sim -K_S$, and there exists a birational morphism $\pi\colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ such that $\pi(C_1)$ is a smooth rational curve of bi-degree $(2, 1)$, $\pi(C_2)$ is a smooth rational curve of bi-degree $(0, 1)$, and $\pi$ is a blow-up of $1 \leq m \leq 4$ distinct points on $\pi(C_1)$ with no point being $\pi(C_1) \cap \pi(C_2)$,*

(III.1) $S \cong \mathbb{P}^2$, *the curves $C_1, C_2, C_3$ are lines,*

(III.2) $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, *$C_1, C_2, C_3$ are of bi-degree $(1, 1), (0, 1)$, and $(1, 0)$, respectively,*

(III.3.n) $S \cong \mathbb{F}_n$ *for any $n \geq 0$, $C_1 = Z_n, C_2 = F$, and $C_3 \in |Z_n + nF|$,*

(III.4.m) *$(S, C)$ is a blow-up of* (III.1) *at $1 \leq m \leq 3$ points in the smooth locus of the boundary curve such that $S$ is a del Pezzo surface, $C_1^2, C_2^2, C_3^2 \geq 0$, i.e., $C_1 + C_2 + C_3 \sim -K_S$, and there exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ such that the curves $\pi(C_1), \pi(C_2), \pi(C_3)$ are lines that have no common intersection, and $\pi$ is a blow-up of $1 \leq m \leq 3$ distinct points on these lines with at most one point on each line and no point on an intersection of two lines,* (III.5.n.m) $(S,C)$ *is a blow-up of* (III.3.n) *at* $m \geq 1$ *points in the smooth locus of the boundary curve such that there exists a birational morphism* $\pi\colon S \to \mathbb{F}_n$ *for some* $n \geq 0$ *such that* $\pi(C_1) = Z_n$, $\pi(C_2) = F$, *and* $\pi(C_3) \in |Z_n + nF|$, *and* $\pi$ *is a blow-up of* $m$ *distinct points on* $\pi(C_1)$ *and* $\pi(C_3)$ *with at most one point on a single curve in the linear system* $|F|$, *and no point being* $\pi(C_1) \cap \pi(C_2)$ *or* $\pi(C_2) \cap \pi(C_3)$,

(IV) $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, *the curves* $C_1$ *and* $C_2$ *are distinct curves of bi-degree* $(1,0)$, *the curves* $C_3$ *and* $C_4$ *are distinct curves of bi-degree* $(0,1)$.

The rest of the section is devoted to the proof of Theorem 3.1.

### 3.1 Basic properties of asymptotically log del Pezzo pairs

In this subsection, before embarking on the proof of Theorem 2.1, we collect several properties of asymptotically log del Pezzo pairs that are not necessarily strongly asymptotically log del Pezzo. These properties are later used in the proof of that theorem, but they should also be useful in a future classification of the former class of pairs.

Thus in the rest of this subsection we assume $(S,C), \beta \in (0,1]^r$, and $r \geq 2$ are as in Definition 1.1.

**Lemma 3.2.** *All curves* $C_1, \ldots, C_r$ *are smooth rational curves.*

*Proof.* Suppose that there exists a non-rational curve among the curves $C_1, \ldots, C_r$. Without loss of generality, we may assume that this curve is $C_1$. Since $-(K_S + \sum_{i=1}^r C_i)$ is nef, it follows from the adjunction theorem that

$$2g(C_1) - 2 + \sum_{i=2}^r C_1.C_i = (K_S + \sum_{i=1}^r C_i).C_1 \leq 0,$$

which implies that $g(C_1) = 1$ and $C_1 \cap C_i = \emptyset$ for every $i \neq 1$. Hence, we see that $C_1$ is an elliptic curve. Arguing as in the proof of Lemma 2.2, we see that there exists an effective divisor $R \sim C_1 + K_S$. Thus, $-R - C_1 - \sum_{i=2}^r C_i \sim -K_S - C$ is nef, implying $R \sim 0$ and $r = 1$, which contradicts our assumption that $r \geq 2$. □

**Lemma 3.3.** *Suppose that there exists a smooth irreducible rational curve* $E$ *on the surface* $S$ *such that* $E^2 = -1$ *and* $E \neq C_i$ *for every* $i$. *Then either* $E$ *is disjoint from* $\sum_{i=1}^r C_i$, *or it intersects exactly one irreducible component of* $\sum_{i=1}^r C_i$. *Moreover, in the latter case* $E$ *intersects that irreducible component transversally at exactly one point.*

*Proof.* Since $-K_S.E = 1$ by adjunction then

$$0 < -(K_S + \sum_{i=1}^r (1-\beta_i)C_i).E = 1 - \sum_{i=1}^r C_i.E + \sum_{i=1}^r \beta_i C_i.E < 2 - \sum_{i=1}^r C_i.E,$$

for small $|\beta|$. This implies that $\sum_{i=1}^r C_i.E < 2$. Hence, either $\sum_{i=1}^r C_i.E = 0$ or $\sum_{i=1}^r C_i.E = 1$, because $E \neq C_i$ for every $i$. In the former case $E \cap C_i = \emptyset$ for every $i$. In the latter case there is an $i$ such that $E.C_i = 1$ and $E \cap C_j = \emptyset$ for every $j \neq i$. □

Similarly to Definition 2.8, let us call the pair $(S, \sum_{i=1}^r C_i)$ *minimal* if there exist no smooth irreducible rational curve on the surface $S$ such that $E^2 = -1$, $E \neq C_i$ for every $i$, and there is a $j$ such that and $E \cap C_j \neq \emptyset$. Then we have the following generalization of Lemma 2.10.

**Lemma 3.4.** *Suppose that there exists a smooth irreducible rational curve $E$ on the surface $S$ such that $E^2=-1$ and $E \neq C_i$ for every $i$. Then there exists a birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface, $\pi(E)$ is a point, the morphism $\pi$ induces an isomorphism $S \setminus E \cong s \setminus \pi(E)$, the divisor $\sum_{i=1}^{r}\pi(C_i)$ is a divisor with simple normal crossings, and $\pi(C_1),\ldots,\pi(C_r)$ are smooth rational curves whose dual graph is the same as the dual graph of the curves $C_1,\ldots,C_r$. Moreover, the pair $(s,\sum_{i=1}^{r}\pi(C_i))$ is asymptotically log del Pezzo and strongly asymptotically log del Pezzo if $(S,C)$ is. Furthermore, if the pair $(S,\sum_{i=1}^{r}C_i)$ is minimal, then the pair $(s,\sum_{i=1}^{r}\pi(C_i))$ is minimal.*

*Proof.* By the Castelnuovo's contractibility criterion, there exists a birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface, $\pi(E)$ is a point, the morphism $\pi$ induces an isomorphism $S\setminus E \cong s\setminus \pi(E)$. Moreover, the divisor $\sum_{i=1}^{r}\pi(C_i)$ is a divisor with a simple normal crossing, the curves $\pi(C_1),\ldots,\pi(C_r)$ are smooth rational curves whose dual graph is the same as the dual graph of the curves $C_1,\ldots,C_r$. Indeed, the latter is obvious if the curve $E$ is disjoint from $\sum_{i=1}^{r}C_i$. If $E$ is not disjoint from $\sum_{i=1}^{r}C_i$, then it intersects exactly one irreducible component of $\sum_{i=1}^{r}C_i$ (and intersects this component transversally and at exactly one point) by Lemma 3.3. The latter implies that the divisor $\sum_{i=1}^{r}\pi(C_i)$ is a divisor with a simple normal crossing, the curves $\pi(C_1),\ldots,\pi(C_r)$ are smooth rational curves whose dual graph is the same as the dual graph of the curves $C_1,\ldots,C_r$. Now we can complete the proof arguing as in the proof of Lemma 2.10 (ii). ☐

The next lemma describes the combinatorial structure of $C$.

**Lemma 3.5.** *(i) Either $|C_i\cap C_j|\leq 1$ for $i\neq j$, or $r=2$, $|C_1\cap C_2|=2$ and $C_1+C_2\sim -K_S$.*
*(ii) If $r\geq 3$, then either the dual graph of the curves $C_1,\ldots,C_r$ forms a tree, or $\sum_{i=1}^{r}C_i\sim -K_S$ and the dual graph of the curves $C_1,\ldots,C_r$ forms a cycle.*
*(iii) If the dual graph of the curves $C_1,\ldots,C_r$ forms a tree, then it is a disjoint union of chains.*

*Proof.* (i) Suppose that $|C_1\cap C_2|\geq 2$. We claim that $r=2$, $C_1+C_2\sim -K_S$, and $|C_1\cap C_2|=2$. By Serre duality,

$$h^2(\mathcal{O}_S(K_S+C_1+C_2))=h^0(\mathcal{O}_S(-C_1-C_2))=0.$$

Put $k=|C_1\cap C_2|$. Then

$$\begin{aligned}(K_S+C_1+C_2).(C_1+C_2)&=(C_1+C_2)^2+K_S.C_1+K_S.C_2\\&=C_1^2+C_2^2+2C_1.C_2+2g(C_1)-2-C_1^2+2g(C_2)-2-C_2^2=2k-4,\end{aligned}$$

since $C_1$ and $C_2$ are rational curves by Lemma 3.2. Since $S$ is rational, it follows from the Riemann–Roch theorem that $h^0(\mathcal{O}_S\big(K_S+C_1+C_2\big))\geq 1+(2k-4)/2=k-1\geq 1$. The rest of the proof is now identical to that of Lemma 3.2.

(ii) By (i), $|C_i\cap C_j|\leq 1$ for every $i\neq j$. Suppose that for some $k\leq r$ the dual graph of the curves $C_1,C_2,\ldots,C_k$ forms a cycle such that $C_k.C_1=C_1.C_2=\ldots=C_{k-1}.C_k=1$, and $C_i.C_j=0$ in all other cases when $1\leq i\neq j\leq k$. We claim that $r=k$. Indeed, as before $h^2(\mathcal{O}_S(K_S+\sum_{i=1}^{k}C_i))=h^0(\mathcal{O}_S(-\sum_{i=1}^{k}C_i))=0$. Since

$$\begin{aligned}(K_S+\sum_{i=1}^{k}C_i).(\sum_{i=1}^{k}C_i)&=2\sum_{1\leq i<j\leq k}C_i.C_j+\sum_{i=1}^{k}K_S.C_j+\sum_{i=1}^{k}C_i^2\\&=2\sum_{1\leq i<j\leq k}C_i.C_j+\sum_{i=1}^{k}(2g(C_i)-2)=2k-2k=0.\end{aligned}$$

Thus, as in (i), $h^0(\mathcal{O}_S(K_S + \sum_{i=1}^k C_i)) \neq 0$ by the Riemann–Roch theorem and there exists an effective divisor $R$ such that $R \sim K_S + \sum_{i=1}^k C_i$, hence the divisor $-K_S - \sum_{i=1}^r C_i \sim -R - \sum_{i=k}^r C_i$ is nef, so $R = 0$ and $r = k$.

(iii) Suppose that the dual graph of the curves $C_1, \ldots, C_r$ forms a tree that is not a disjoint union of chains. Then $r \geq 4$, and there exists a curve among $C_1, \ldots, C_r$ that intersects at least three other different curves among $C_1, \ldots, C_r$, say $C_1.C_2 = 1$, $C_1.C_3 = 1$, and $C_1.C_4 = 1$. Then

$$
\begin{aligned}
0 > (K_S + \sum_{i=1}^r (1-\beta_i)C_i).C_1 &= K_S.C_1 + (1-\beta_1)C_1^2 + \sum_{i=2}^4 (1-\beta_i)C_i.C_1 + \sum_{i=5}^r (1-\beta_i)C_i.C_1 \\
&\geq K_S.C_1 + (1-\beta_1)C_1^2 + \sum_{i=2}^4 (1-\beta_i)C_i.C_1 \\
&= -2 + C_1^2 + (1-\beta_1)C_1^2 + \sum_{i=2}^4 (1-\beta_i)C_i.C_1 = 1 - \beta_1 C_1^2 - \beta_2 - \beta_3 - \beta_4 > 0,
\end{aligned}
$$

for $|\beta| \ll 1$, a contradiction. $\square$

The next lemma shows that only curves $C_i$ that are at the 'tail' of a chain can have negative self-intersection.

**Lemma 3.6.** *Suppose that $(S, \sum_{i=1}^r C_i)$ is strongly asymptotically log del Pezzo. Then $C_i^2 \geq 0$ for every $C_i$ such that $C_i$ intersects at least two curves among $C_1, \ldots, C_r$ different from itself. Similarly, $C_i^2 \geq 0$ if there exists a curve among $C_1, \ldots, C_r$ different from $C_i$ that intersects $C_i$ by more than one point.*

*Proof.* Suppose that $C_1$, say, intersects at least two curves among $C_2, \ldots, C_r$, say $C_1.C_2 = C_1.C_3 = 1$. Suppose that $C_1^2 < 0$. Then it follows from adjunction that

$$
\begin{aligned}
(K_S + \sum_{i=1}^r (1-\beta_i)C_i).C_1 &\geq K_S.C_1 + (1-\beta_1)C_1^2 + (1-\beta_2)C_2.C_1 + (1-\beta_3)C_3.C_1 \\
&= -\beta_1 C_1^2 - \beta_2 - \beta_3
\end{aligned}
$$

thus $\beta_1 < \beta_2 + \beta_3$. The latter contradicts our assumption that the divisor $-(K_S + \sum_{i=1}^r (1-\beta_i)C_i)$ is ample for every $(\beta_1, \ldots, \beta_r) \in (0,1]^r$ with $|(\beta_1, \ldots, \beta_r)| < \epsilon$.

To complete the proof, we may assume that $C_1$ intersects some curve among $C_2, \ldots, C_r$ by more than 2 points. Then $r = 2$ and $C_1.C_2 = 2$ by Lemma 3.5. If $C_1^2 < 0$, then

$$
0 > (K_S + \sum_{i=1}^r (1-\beta_i)C_i).C_1 = K_S.C_1 + (1-\beta_1)C_1^2 + (1-\beta_2)C_2.C_1 = -\beta_1 C_1^2 - 2\beta_2
$$

which once again does not hold for all small $\beta$. $\square$

*Remark* 3.7. We mention that the number of connected components of the curve $\sum_{i=1}^r C_i$ is at most 2 [40, Theorem 6.9] (see also [15, Proposition 2.1] for a generalization in all dimensions). We will not use this result in the proof of Theorem 3.1. In fact, if $(S, \sum_{i=1}^r C_i)$ is strongly asymptotically log del Pezzo, this also follows from Theorem 3.1.

The next example shows that the previous lemma does not hold in the non-strongly asymptotically log del Pezzo regime (nor in the "diagonal regime", i.e., where $\beta = \beta_1(1, \ldots, 1)$), where 'interior' boundary components could have negative self-intersection.

**Example 3.8.** Let $S \cong \mathbb{F}_n$ for some $n > 0$. Let $C_1$ and $C_2$ be two distinct fibers of the natural projection $\mathbb{F}_n \to \mathbb{P}^1$, and let $C_3 = Z_n$. Then the pair $(S, \sum_{i=1}^3 C_i)$ is asymptotically log Fano, but it is not strongly asymptotically log Fano. Indeed, by (1.9) we see that the divisor $-K_S - \sum_{i=1}^3 (1-\beta_i)C_i)$ is ample if and only if $\beta_1 + \beta_2 > n\beta_3$.

## 3.2 Classification

Note that in §3.1 we only assumed that $(S, \sum_{i=1}^r C_i)$ is asymptotically log del Pezzo. In this subsection, we assume that $(S, \sum_{i=1}^r C_i)$ is strongly asymptotically log del Pezzo (Definition 1.1). Namely, there exists a positive $\epsilon \in (0,1]$ such that the divisor

$$- K_S - \sum_{i=1}^{r} (1-\beta_i)C_i \tag{3.1}$$

is ample for every $\beta = (\beta_1, \dots, \beta_r) \in (0,1]^r$ with $|\beta| \leq \epsilon$.

### 3.2.1 Boundary with arithmetic genus one

**Lemma 3.9.** *Suppose that $\sum_{i=1}^r C_i \sim -K_S$. Then $-K_S$ is ample, $C_i \cong \mathbb{P}^1$ $\forall i$, and $C_i^2 \geq 0$ $\forall i$. Furthermore, if $r = 2$, then $|C_1 \cap C_2| = 2$. If $r \geq 3$, then $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and the dual graph of the curves $C_1, \dots, C_r$ forms a cycle.*

*Proof.* The ampleness of $-K_S$ is obvious, because $(S, \sum_{i=1}^r C_i)$ is asymptotically log del Pezzo

$$-(K_S + \sum_{i=1}^{r} (1-\beta)C_i) \sim_{\mathbb{R}} -\beta K_S$$

for every real $\beta \in (0,1]$. By Lemmas 3.2, $C_i \cong \mathbb{P}^1$ $\forall i$. By Lemma 3.6, $C_i^2 \geq 0$ $\forall i$. If $r = 2$, then $|C_1 \cap C_2| = 2$ by Lemma 3.5. Similarly, if $r \geq 3$, then it follows from Lemma 3.5 that $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and the dual graph of the curves $C_1, \dots, C_r$ forms a cycle. □

In analogy with Corollary 2.3, we get:

**Corollary 3.10.** *Suppose $\sum_{i=1}^r C_i \sim -K_S$. Then $(S, \sum_{i=1}^r C_i)$ is one of* (II.1A), (II.4A), (II.5A.m), (II.8.m), (III.1), (III.2), (III.3.m), *or* (IV).

### 3.2.2 Boundary with arithmetic genus zero

To complete the proof of the classification part of Theorem 3.1, we may assume that $\sum_{i=1}^r C_i \not\sim -K_S$. By Lemma 3.5 $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and the dual graph of the curves $C_1, \dots, C_r$ is a union of disjoint chains. By Lemma 3.6, $C_k^2 \geq 0$ for every curve $C_k$ among $C_1, \dots, C_r$ that intersects at least 2 other different curves among the curves $C_1, \dots, C_r$. The next lemma gives a complete classification in this situation under the further assumption that the Picard group is small.

**Lemma 3.11.** *Suppose that* $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$ *and $C \not\sim -K_S$. Then when $(S, C)$ is minimal it is one of* (II.1B), (II.2A.n), (II.2B.n), (II.2C.n), (II.3), *or* (III.3.n), *and otherwise it is* (II.5B.1).

*Proof.* Since $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$, either $S \cong \mathbb{P}^2$ or $S \cong \mathbb{F}_n$ for some $n \geq 0$. If the latter case $(S, C)$ must be (II.1B), as $C \not\sim -K_S$. Assume from now on that $S = \mathbb{F}_n$.

Recall that $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and $C_1, \dots, C_r$ are smooth rational curves whose dual graph is a union of disjoint chains. If $n = 0$ this determines $(S, C)$, i.e., the boundary $C$ must be either two disjoint fibers (II.2A.0), two intersecting fibers (II.2C.0), a fiber and a bi-degree (1,1) curve (II.2B.0), or three non-disjoint fibers (III.3.0).

To complete the proof let us first consider the case $n \geq 2$. First,

$$0 \leq -\Big(K_S + \sum_{i=1}^{r}(1-\beta_i)C_i\Big).Z_n = 2 - n - \sum_{i=1}^{r}(1-\beta_i)C_i.Z_n, \tag{3.2}$$

so one of the curves, say $C_1$, equals $Z_n$. If every curve $C_2, \dots, C_r$ lies in $|F|$, then

$$0 < -\Big(K_S + \sum_{i=1}^{r}(1-\beta_i)C_i\Big).Z_n = -n\beta_1 + 2 - \sum_{i=2}^{r}(1-\beta_i),$$

thus $r = 2$ and $(S, C)$ is (II.2C.n). Assume that $C_2 \not\sim F$ and write $C_i \sim a_i Z_n + b_i F$. Then since $[F]$ is nef

$$0 < -\Big(K_S + \sum_{i=1}^{r}(1-\beta_i)C_i\Big).F = 1 + \beta_1 - a_2(1-\beta_2) - \sum_{i=3}^{r}(1-\beta_i)C_i.F \geq 1 + \beta_1 - a_2(1-\beta_2),$$

and since $a_2 > 0$ this implies that $a_2 = 1$. Then

$$0 < -\Big(K_S + \sum_{i=1}^{r}(1-\beta_i)C_i\Big).F = \beta_1 + \beta_2 - \sum_{i=3}^{r}(1-\beta_i)C_i.F,$$

i.e., $C_i.F = 0$ for every $i \geq 3$. Therefore, we see that $C_i \in |F|$ for every $i \geq 3$. Then

$$-\Big(K_S + \sum_{i=1}^{r}(1-\beta_i)C_i\Big) \sim (\beta_1 + \beta_2)Z_n + (n + 2 - b_2 - \sum_{i=3}^{r}(1-\beta_i))F,$$

where $b_2 \geq n$ by (1.10). Then by (1.9)

$$n + 2 - b_2 - \sum_{i=3}^{r}(1-\beta_i) > n(\beta_1 + \beta_2).$$

If $r = 2$ this implies that $b_2 \leq n+1$ so $b_2 \in \{n, n+1\}$, i.e., $(S, C)$ is (II.2A.n) or (II.2B.n) and if $r = 3$ then $b_2 \leq n$ so $b_2 = n$ so $(S, C)$ is (III.3.n).

Finally, assume $n = 1$. Then (3.2) implies that either $C_1 = Z_1$ or $C_1 \sim Z_1 + F$. In the former case the same arguments of the previous paragraph apply to yield $(S, C)$ is either (II.2C.1), (II.2A.1), (II.2B.1), or (III.3.1). In the latter case, if $C_2 \sim F$ then $r = 2$ and $(S, C)$ is (II.5B.1) and is not minimal since $Z_1$ intersects $C_2$ transversally at one point. Due to (3.2) the only other remaining possibility is $C_2 \sim Z_1 + F$ and then $(S, C)$ is (II.3). The proof is now complete since all the cases listed in the statement are indeed strongly asymptotically log del Pezzo by §4.2. $\square$

The following is an analogue of Lemma 3.4 for the case when a $-1$-curve contained in the boundary is contracted. We omit the proof as it is analogous to the proof of that lemma.

Recall that $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and the dual graph of the curves $C_1, \dots, C_r$ is a union of disjoint chains.

**Lemma 3.12.** *Suppose that $C_1^2 = -1$. Then there exists a birational morphism $\pi\colon S \to s$ such that $s$ is a smooth surface, $\pi(C_1)$ is a point, the morphism $\pi$ induces an isomorphism $S \setminus C_k \cong s \setminus \pi(C_1)$, the divisor $\sum_{i=2}^r \pi(C_i)$ is a divisor with simple normal crossings, $|\pi(C_i) \cap \pi(C_j)| \leq 1$ for every $i \neq j$, and $\pi(C_2), \ldots, \pi(C_r)$ are smooth rational curves whose dual graph is a union of disjoint chains. Moreover, the pair $(s, \sum_{i=2}^r \pi(C_i))$ is strongly asymptotically log del Pezzo. Furthermore, if the pair $(S, \sum_{i=1}^r C_i)$ is minimal, then the pair $(s, \sum_{i=2}^r \pi(C_i))$ is minimal as well.*

*Proof.* Recall that $|C_i \cap C_j| \leq 1$ for every $i \neq j$, and $C_1, \ldots, C_r$ are smooth rational curves whose dual graph is a union of disjoint chains. Moreover, it follows from Lemma 3.6 that $C_1^2 \geq 0$ if $C_1$ intersects at least 2 curves among the curves $C_2, \ldots, C_r$. Arguing as in the proof of Lemma 3.4, we obtain all required assertions. $\square$

Now we are ready to prove an analogue of Lemma 2.11 that plays a crucial role in the proof of Theorem 3.1.

**Lemma 3.13.** *Suppose that $(S, \sum_{i=1}^r C_i)$ is minimal. Then* $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$.

*Proof.* If $C \sim -K_S$ then $(S, C)$ is one of the pairs listed in Corollary 3.10. Of those, (II.1A), (II.4A), (III.1), (III.2), and (IV) are minimal and they all satisfy $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$. So we assume from now on that $C \not\sim -K_S$.

Suppose that $\mathrm{rk}(\mathrm{Pic}(S)) \geq 3$. Let us derive a contradiction. By (1.11) there exists a smooth rational curve $E$ on the surface such that $E^2 = -1$. Either $E \neq C_i$ for every $i$, or there is $k$ such that $E = C_k$. By Lemmas 3.4 and 3.12 and induction on $\mathrm{rk}(\mathrm{Pic}(S))$ we can assume that $\mathrm{rk}(\mathrm{Pic}(S)) = 3$.

If $E \neq C_i$ for every $i$, then we can proceed exactly as in the proof of Lemma 2.11 to obtain a contradiction. Thus, assume that $E = C_1$. By Lemma 3.6, $C_1$ intersects at most one curve among the curves $C_1, \ldots, C_r$. Suppose that $C_1 \cap C_i = \emptyset$ for every $i \geq 3$ (if any).

Since the pair $(S, C)$ is minimal and strongly asymptotically log del Pezzo, there exists a birational morphism $\pi\colon S \to s$ as in Lemma 3.12. Then $(s, \sum_{i=2}^r \pi(C_i))$ is minimal and $\mathrm{rk}(\mathrm{Pic}(s)) = 2$, and, in particular, $s \cong \mathbb{F}_n$ for some $n \geq 0$.

Put $c_i = \pi(C_i)$ for every $i \geq 2$. Let $\xi\colon S \to \mathbb{P}^1$ be the natural projection (it is uniquely determined if $n \neq 0$). Then either

$$\pi(C_1) \notin \bigcup_{i=2}^r c_i,$$

(if $C_1 \cap C_2 = \emptyset$) or $\pi(C_1) \in c_2$ and $\pi(C_1) \notin c_i$ for every $i \geq 3$ (if any) (if $C_1 \cap C_2 \neq \emptyset$).

We can apply Lemmas 2.9 and 3.11 to get an explicit description of the pair $(s, \sum_{i=2}^r c_i)$. The cases (I.1B), (I.1C), (I.6B.1), (I.6C.1), (II.1B), and (II.5B.1) are excluded because either the rank of their Picard group is one or else they are not minimal. Thus, if $r \geq 2$, $(s, \sum_{i=2}^r c_i)$ is one of (II.2A.n), (II.2B.n), (II.2C.n), (II.3), or (III.2.n). Similarly, if $r = 1$, $(s, \sum_{i=2}^r c_i)$ is one of (I.2.n), (I.3A), (I.3B), (I.4B), or (I.4C). In particular, $r$ is at most four.

Let $f$ be a fiber of the morphism $\xi$ that passes through the point $\pi(C_1)$, and let $F$ be its proper transform on $S$. Then $F$ is a smooth irreducible rational curve such that $F^2 = -1$. Moreover, we have $F \cap C_1 \neq \emptyset$ by construction. Since $(S, \sum_{i=1}^r C_i)$ is minimal, the curve $F$ must be one of the curves $C_2, \ldots, C_r$. Then $F = C_2$, $C_1 \cap C_2 \neq \emptyset$, $\pi(C_1) \in c_2$, and $\pi(C_1) \notin c_i$ for every $i \geq 3$ (if any). Moreover, it follows from Lemma 3.6 that $C_2$ does not intersect any curve among $C_3, \ldots, C_r$ (if any), since $F^2 = -1$. Thus, $c_2$ does not intersect any curve among $c_3, \ldots, c_r$ (if any).

Suppose that $r = 2$. Then $(s, c_2)$ is one of (I.2.n), (I.3A), (I.3B), (I.4B), or (I.4C). The latter is possible only in the case (I.2.0) since $c_2 \in |f|$ is a fiber of the the morphism $\xi$. Then $s \cong \mathbb{P}^1 \times \mathbb{P}^1$. The latter implies that $(S, C_1 + C_2)$ is not minimal. Indeed, the surface $S$ is a del Pezzo surface with $K_S^2 = 7$. It contains three $(-1)$-curves. Two of them are the curves $C_1$ and $C_2$. The third one intersects $C_2$, contradicting minimality.

Suppose that $r = 3$. As noted earlier then $(s, c_2+c_3)$ is one of (II.2A.n), (II.2B.n), (II.2C.n), or (II.3). Since $c_2 \cap c_3 = \emptyset$ it must be (II.2A.n). Since $c_2$ is a fiber of the morphism $\xi$ and $c_2 \cap c_3 = \emptyset$ it follows that $c_3$ is also a fiber of $\xi$. Thus, $n = 0$, i.e., $s \cong \mathbb{P}^1 \times \mathbb{P}^1$. The latter implies that $(S, C_1+C_2)$ is not asymptotically log del Pezzo. Indeed, the surface $S$ is del Pezzo surface with $K_S^2 = 7$. It contains three $(-1)$-curves. Two of them are the curves $C_1$ and $C_2$. The third one intersects $C_2$ and $C_3$, contradicting Lemma 3.3.

Suppose that $r = 4$. Then $(s, c_2 + c_3 + c_4)$ must be (III.2.n). But this is precluded by the fact that $c_2$ does not intersect $c_3$ or $c_4$.

In conclusion then $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$. □

We now complete the proof of the classification part of Theorem 3.1. If $\mathrm{rk}(\mathrm{Pic}(S)) \leq 2$, then $(S, C)$ is listed in Corollary 3.10 if $C \sim -K_S$ and by Lemma 3.11 if $C \not\sim -K_S$. On the other hand, if $\mathrm{rk}(\mathrm{Pic}(S)) > 2$, the pair $(S, \sum_{i=1}^r C_i)$ is not minimal by Lemma 3.13. But then, Lemmas 3.4 and 3.12 imply $(S, C)$ is a blow-up along the boundary of one of the minimal pairs that we already listed. It remains to check the genericity conditions on the location of the blow-up points as stated in Theorem 3.1. This is carried out in §4.2 where we simultaneously also verify that all pairs listed in the theorem are indeed strongly asymptotically del Pezzo.

*Remark* 3.14. The results of this section already give some hints as to the difficulties in classifying all asymptotically log del Pezzo surfaces. In particular, $r$ can then be infinite, and $-1$-curves can appear as 'interior' curves of the boundary (see Example 3.8) even though the number of connected components of the support of $C$ is still two by Remark 3.7. However, the classification of the 'diagonal' regime (where $-K_S - \sum(1-\beta_i)C_i$ is ample for all sufficiently small $\beta$ of the form $\beta = \beta_1(1, \ldots, 1)$) should be more tractable. In a related vein, results of di Cerbo–di Cerbo [11] give bounds on the largest possible value $\beta_1$ may take for pairs of class (ד) in this last regime depending only on $(K_S+C)^2$. We further note that di Cerbo [12] considered the 'diagonal' regime in the setting of negative curvature, and obtained necessary and sufficient intersection-theoretic restrictions on the pair for $K_S + (1-\beta)\sum C_i$ to be ample. Of course, in the negative setting a complete classification is lacking even in the smooth setting with no boundary.

# 4 Positivity properties of the logarithmic anticanonical bundle

Let $S$ be a smooth surface, let $C_1, \ldots, C_r$ be smooth irreducible curves on the surface $S$ such that $\sum_{i=1}^r C_i$ is a divisor with simple normal crossings, and let $\beta = (\beta_1, \beta_2, \ldots, \beta_r) \in (0,1]^r$, where $r \geq 1$. Suppose that $(S, \sum_{i=1}^r C_i)$ is strongly asymptotically log del Pezzo. Then we have the following mutually excluding possibilities:

(א) $-(K_S + \sum_{i=1}^r C_i) \sim 0$, $S$ is del Pezzo surface, and $\sum_{i=1}^r C_i \sim -K_S$, and $C_i^2 \geq 0\ \forall i$,

(ב) $(-K_S - \sum_{i=1}^r C_i)^2 = 0$, all curves $C_1, \ldots, C_r$ are rational, and the dual graph of the curves $C_1, \ldots, C_r$ is a disjoint union of chains,

(ג) the divisor $-(K_S+\sum_{i=1}^r C_i)$ is big and nef, the divisor $-(K_S+\sum_{i=1}^r C_i)$ is not ample, all curves $C_1,\ldots,C_r$ are rational and the dual graph of the curves $C_1,\ldots,C_r$ is a disjoint union of chains,

(ד) $C\cong\mathbb{P}^1$, the divisor $-(K_S+\sum_{i=1}^r C_i)$ is ample, all curves $C_1,\ldots,C_r$ are rational and the dual graph of the curves $C_1,\ldots,C_r$ is a disjoint union of chains.

This follows from Lemmas 2.12, 2.2, 3.6 and 3.5. Note that the classification in Theorems 2.1 and 3.1 generalizes Maeda's classical result [33] that corresponds to class (ד). Maeda further provided a full classification in class (ד). In §4.1 we generalize this by giving a complete classification in each of the remaining classs. In §4.3 we go further in the class (ב) by proving that the linear system $|-K_S-\sum_{i=1}^r C_i|$ gives a morphism $S\to\mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$.

*Remark* 4.1. There is perhaps no real need to distinguish between classes (ג) and (ד), but we do that mainly for a historical reason. Indeed, class (ד) is not new. These pairs were completely classified in Maeda's work who coined the term 'log del Pezzo surface' for this class of pairs [33]. In higher dimensions it might prove more natural to identify only $\dim X+1$ classes according to the Kodaira dimension of $-K_X-D$.

## 4.1 Positivity classification

We now prove Theorem 1.4, relying on Theorems 2.1 and 3.1.

Class (א) follows from Corollaries 2.3 (and the remark preceeding it) and 3.10. Class (ד) follows from (1.9).

Next, if $(S,\sum_{i=1}^r C_i)$ is not minimal (see Definition 2.8), then it follows from the proof of Theorems 2.1 and 3.1 that there exists a non-biregular birational map $\pi\colon S\to s$ such that the pair $(s,\sum_{i=1}^r\pi(C_i))$ is minimal and

$$-K_S+\sum_{i=1}^r C_i\sim\pi^*(-K_s-\sum_{i=1}^r\pi(C_i)).$$

Indeed, by our construction, each $-1$-curve that is contracted intersects the boundary transversally exactly at one point. This also shows that $-K_S-\sum_{i=1}^r C_i$ can not be ample if $(S,\sum_{i=1}^r C_i)$ is not minimal, because $-K_S-\sum_{i=1}^r C_i$ intersects all $\pi$-exceptional curves trivially. In sum, if $(S,C)$ is of class (א), respectively (ב), then so is $(s,c)$, and if $(S,C)$ is of class (ג) or (ד) $(s,c)$ is of class (ג). This completes the verification of class (ג) since each of these pairs are blow-ups of a pair of class (ד), while the pairs (I.3A), (I.4B), (II.2A.n), (II.2B.n), (II.3), and (III.3.n) all satisfy $(K_S+C)^2=0$. Class (ב) then contains, by exclusion, all the remaining pairs listed in Theorems 2.1 and 3.1.

## 4.2 Verification of the list

Using the positivity classification of the original lists of Theorems 2.1 and 3.1, we now verify that indeed each of the pairs listed there is strongly asymptotically log del Pezzo. This is the last step remaining to complete the proof of the main classification result, Theorem 1.2.

The Maeda case (ד) is immediate by convexity as then $-K_S-C$ itself is ample, and so is the case (א). So suppose $(S,C)$ is a pair of class (ב) or (ג) listed in Theorem 1.4. Then there exists a blow-down map $\pi:S\to s$ such that the pair $(s,c)$ is minimal where $c=\pi(C)$. Then

$$-K_S-(1-\beta)C\equiv\pi^*\Big(-\big(K_s+(1-\beta)c\big)\Big)-\sum_{i=1}^m\beta E_i. \tag{4.1}$$

Here $E_i = \pi^{-1}(P_i)$, with $P_1, \dots, P_m$ denoting the blow-up points. The slight subtlety is that while the second term on the right is 'small' in terms of its contribution to intersection numbers and the first term is ample, the latter also depends on $\beta$ and so a priori it is not clear which term will dominate. In fact, the following example illustrates a situation where such a problem arises.

**Example 4.2.** Consider the surface $\mathbb{F}_n$ and let $R$ be some smooth curve in $|Z_n + nF|$. Then $(\mathbb{F}_n, Z_n + F + R)$ is strongly asymptotically log Fano. Let $\pi\colon S \to \mathbb{F}_n$ be a blow-up of $m$ distinct points in the smooth locus of $Z_n + F + R$ such that no two of the points lie on one curve in the linear system $|F|$. Let $C_1$, $C_2$, $C_3$ be the proper transforms of the curves $Z_n$, $F$, $R$, respectively. Then $(S, C_1 + C_2 + C_3)$ is asymptotically log Fano. On the other hand, $(S, C_1 + C_2 + C_3)$ is strongly asymptotically log Fano if and only if none of the blow-up points lies in $F$ (cf. (III.5.n.m) in Theorem 3.1).

We now go through the lists of classes $\beth$ and $\gimel$ and verify the pairs are strongly asymptotically log del Pezzo. We assume without mention that $\beta$ is taken in each equation to be sufficiently small, depending only on $(S, C)$.

By the Nakai–Moishezon criterion (1.4), we have to check that $(K_S + (1-\beta)C)^2 > 0$ and $-(K_S + (1-\beta)C).Z > 0$ for every irreducible curve $Z \subset S$, with $\beta$ independent of $Z$ (and we use, e.g., the notation $O_{n,m}(\beta_1)$ to denote a quantity bounded by $C\beta_1$ with $C$ depending only on $n, m$, and $(S, C)$).

To do this, let us fix some irreducible curve $Z$ on the surface $S$. We may assume that $Z$ is not $\pi$-exceptional since by (4.1) and the fact that all the blow-up points are distinct (so none of the exceptional divisors intersect) $-(K_S + (1-\beta)C)Z > 0$ if $Z$ is $\pi$-exceptional. Put $z := \pi(Z)$, and suppose that $\mathrm{mult}_{P_1}(z) \le \cdots \le \mathrm{mult}_{P_m}(z)$.

**Class ($\gimel$).** Suppose that we are either in case (I.6B.m) or (I.6C.m). Then $S \cong \mathbb{P}^2$, and $c$ is a conic in the case (I.6B.m) or a line in the case (I.6C.m). Let $\delta$ be the degree of the curve $c$ in $S \cong \mathbb{P}^2$, i.e., either $\delta = 2$ in the case (I.6B.m) or $\delta = 1$ in the case (I.6C.m). Let $l$ be a line in $S$. Then $-(K_S + (1-\beta)C) \equiv \pi^*((3 - (1-\beta)\delta)l) - \sum_{i=1}^m \beta E_i$, which implies that $(K_S + (1-\beta)C)^2 = (3 - \delta + \delta\beta)^2 - m\beta^2 > 0$ since $\delta \in \{1, 2\}$. Let $d$ be the degree of the curve $z$ in $S \cong \mathbb{P}^2$. Then $-(K_S + (1-\beta)C).Z = (3 - (1-\beta)\delta)d - \sum_{i=1}^m \beta \mathrm{mult}_{P_i}(z) \ge d - m\beta \mathrm{mult}_{P_m}(z) > 0$, concluding these cases.

Now consider the case (I.7.n.m). Then $s \cong \mathbb{F}_n$, and $c = Z_n$. Let $f$ be a fiber of the natural projection $s \to \mathbb{P}^1$, i.e., $f$ is a smooth irreducible rational curve such that $f.c = 1$ and $f^2 = 0$. Then $-K_s \sim 2c + (2+n)f$, so $-(K_S + (1-\beta)C) \equiv \pi^*((1+\beta)c + (2+n)f) - \sum_{i=1}^m \beta E_i$ and $(K_S + (1-\beta)C)^2 = 4 + n + 4\beta - n\beta^2 - m\beta^2 > 0$. Note that $z \sim a_1 c + a_2 f$ for some non-negative integers $a_1, a_2$ such that either $(a_1, a_2) = (1, 0)$ (if $n \ge 1$, then $Z = C$ in this case), or $(a_1, a_2) = (0, 1)$ (this means that $z$ is a fiber of the projection $s \to \mathbb{P}^1$), or $a_2 \ge n a_1$ (see [20, Corollary 2.18]). Then $-(K_S + (1-\beta)C).Z = a_2 + 2a_1 + \beta(a_2 - n a_1) - \sum_{i=1}^m \beta \mathrm{mult}_{P_i}(z) > 0$. Indeed, since the divisor $c + (n+1)f$ is very ample [20, Theorem 2.17], we get a uniform bound on the multiplicity:

$$\mathrm{mult}_{P_m}(z) \le (c + (n+1)f)z = a_2 + a_1.$$

This concludes this case.

The case (I.8B.m) is treated similarly.

Suppose now $(S, C)$ is (I.9C.m). Then $s \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $f_1, f_2$ be fibers of the two natural projections $s \to \mathbb{P}^1$. Then $c \sim f_1 + f_2$ and $-K_s \sim 2f_1 + 2f_2$, and $z \sim a_1 f_1 + a_2 f_2$ for some non-negative integers $a_1$ and $a_2$ such that $(a_1, a_2) \ne (0, 0)$. Then $-K_S - (1-\beta)C = \pi^*((1 + \beta)f_1 + (1+\beta)f_2) - \sum_{i=1}^m \beta E_i$, and $(K_S + (1-\beta)C)^2 = 2(1+\beta)^2 - m\beta^2 > 0$. Since $f_1 + f_2$ is very ample $\mathrm{mult}_{P_m}(z) \le a_1+a_2$. Thus, $-(K_S+(1-\beta)C).Z = (a_1+a_2)(1+\beta) - \sum_{i=1}^m \beta \mathrm{mult}_{P_i}(z) \ge (a_1+a_2)(1+\beta) - m\beta \mathrm{mult}_{P_m}(z) > 0$.

Finally, one readily checks that the case (II.5B.m) reduces to (II.1B) which in turns is essentially identical to (I.1B). Similarly, (II.6C.n.m) reduces to (II.2C.n) which is in the class ($\daleth$).

**Class ($\beth$).** The cases (I.3A) and (I.4B) are immediate. Suppose we are in the case (I.9B.m). Then $-K_S-(1-\beta)C \equiv \pi^*(2\beta f_1+(1+\beta)f_2) - \sum_{i=1}^m \beta E_i$, so $(K_S+(1-\beta)C)^2 = 4\beta(1+\beta)-m\beta^2 > 0$, and

$$-(K_S+(1-\beta)C).Z = 2\beta a_2 + (1+\beta)a_1 - \sum_{i=1}^m \beta \mathrm{mult}_{P_i}(z) > 0 \tag{4.2}$$

since if $z \in |f_2|$ (i.e., $(a_1,a_2)=(0,1)$) then $z$ passes through at most one of the blow-up points and in this case $z$ (a fiber) is also necessarily smooth, so $\mathrm{mult}_{P_i}(z) \in \{0,1\}$.

Among (II.2A.n) and (II.2B.n) it suffices to check the latter. In fact, since (II.6A.n.m) and (II.6B.n.m) are their blow-ups we only need to consider (II.6B.n.m) (allowing $m$ to possibly equal 0). In this case, $-K_S-(1-\beta)C = \pi^*\big((\beta_1+\beta_2)Z_n + (1+(n+1)\beta_2)F\big) - \sum_{i=1}^k \beta_1 E_i - \sum_{i=k+1}^m \beta_2 E_i$, assuming that exactly the first $k$ points are blown-up along $\pi(C_1) = Z_n$. The square of this class is then $2\beta_1+2\beta_2+O_{n,m}(\beta_1\beta_2+\beta_1^2+\beta_2^2) > 0$, and its intersection with $Z$ (such that $z \sim a_1Z_n+a_2F$) equals $-na_1(\beta_1+\beta_2)+a_1(1+(n+1)\beta_2)+a_2(\beta_1+\beta_2)-\beta_1\sum_{i=1}^k \mathrm{mult}_{P_i}(z) - \beta_2\sum_{i=k+1}^m \mathrm{mult}_{P_i}(z) = a_1(1+b_2-n\beta_1)+a_2(\beta_1+\beta_2)-\beta_1\sum_{i=1}^k \mathrm{mult}_{P_i}(z)-\beta_2\sum_{i=k+1}^m \mathrm{mult}_{P_i}(z)$. This is positive if $a_1>0$ since, as before, the multiplicities are uniformly bounded independently of $z$. If $a_1=0$ then $a_2>0$, so $a_2=1$ as $z$ is irreducible, thus a fiber. Then the intersection number is positive (bounded below by $\min\{\beta_1,\beta_2\}$) provided the fiber does not intersect more than one of the $P_i$.

The case (II.7.m) (that implies the case (II.3)) is proven using very similar computations.

Finally we consider (III.5.n.m) (that takes care of the case (III.3.n)). Then $-K_S-(1-\beta)C = \pi^*\big((\beta_1+\beta_3)Z_n+(1+\beta_2+n\beta_3)F\big) - \sum_{i=1}^k \beta_1E_i - \sum_{i=k+1}^m \beta_3E_i$ This squares to $-n(\beta_1+\beta_3)^2 + 2(\beta_1+\beta_3)(1+\beta_2+n\beta_3) - k\beta_1^2 - (m-k-1)\beta_3^2 > 0$ (here we see why blow-ups along $\pi(C_2)$ are prohibited). The verification of the intersection with $Z$ is as in the previous case.

The proof of Theorem 1.2 is now complete.

### 4.3 Nef and non-big adjoint anticanonucal bundle

In the case ($\beth$), the linear system $|-(K_S+\sum_{i=1}^r C_i)|$ gives a morphism $S \to \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$. This can be shown by using our classification in Theorems 2.1 and 3.1 or alternatively (and in any dimension) from Kawamata–Shokurov's results as demonstrated in Theorem 1.9. But we prefer to give a self-contained classification-free proof of Proposition 1.7 that does not rely on these deep works.

In the remaining part of this subsection we prove Proposition 1.7. Suppose that $(K_S + \sum_{i=1}^r C_i)^2 = 0$. By Lemma 3.5, the dual graph of the curves $C_1,\ldots,C_r$ is a disjoint union of chains. Let $l$ be the number of connected components of the curve $\sum_{i=1}^r C_i$ (by Remark 3.7 one has $l \le 2$ but we will not use it here).

**Lemma 4.3.** *One has $h^0(\mathcal{O}_S(-(K_S+\sum_{i=1}^r C_i))) = 1+l$.*

*Proof.* Since the dual graph of the curves $C_1,\ldots,C_r$ is a disjoint union of chains, one can easily check that

$$(K_S+\sum_{i=1}^r C_i).(\sum_{i=1}^r C_i) = -2l.$$

This allows us to compute $h^0(\mathcal{O}_S(-(K_S+\sum_{i=1}^r C_i)))$. Indeed, we have

$$h^2(\mathcal{O}_S(-(K_S+\sum_{i=1}^r C_i))) = h^1(\mathcal{O}_S(-(K_S+\sum_{i=1}^r C_i))) = 0$$

by (3.1) and the Kawamata–Viehweg Vanishing Theorem. Therefore, it follows from the Riemann–Roch Theorem that

$$\begin{aligned} h^0\Big(\mathcal{O}_S\big(-(K_S+\sum_{i=1}^r C_i)\big)\Big) &= 1+\frac{(K_S+\sum_{i=1}^r C_i).(2K_S+\sum_{i=1}^r C_i)}{2} \\ &= 1-(K_S+\sum_{i=1}^r C_i).(\sum_{i=1}^r C_i) = 1+l, \end{aligned}$$

because $(-K_S-\sum_{i=1}^r C_i)^2 = 0$ by assumption. □

Thus, we see that $|-(K_S+C)|$ is at least a pencil. Moreover, if $l=1$, then it is a pencil, since $S$ is rational. Note that we can use [40, Theorem 6.9] to show that $l \leq 2$. But we do not need this. In fact, one can show that $l \leq 2$ using Lemma 4.3 (cf. the proof of [40, Theorem 6.9]).

**Lemma 4.4.** *The linear system $|-K_S-\sum_{i=1}^r C_i|$ is a base point free.*

*Proof.* Let us first show that $|-(K_S+C)|$ is free from fixed components (see [19, Theorem III.1]). Suppose this is not the case. Let $B$ be the fixed part of the linear system $|-(K_S+C)|$, and let $M$ be its mobile part. Then $M$ is nef. In particular, we have $h^1(\mathcal{O}_S(M)) = h^2(\mathcal{O}_S(M)) = 0$ by the Kawamata-Viehweg vanishing theorem. Then it follows from the Riemann–Roch theorem that

$$2l = h^0(\mathcal{O}_S(M+B) = h^0(\mathcal{O}_S(M) = 1+\frac{M.(M-K_S)}{2},$$

which implies that $M^2-M.K_S = 4l-2$. On the other hand, we have

$$\begin{aligned} 0 = (K_S+\sum_{i=1}^r C_i)^2 = (B+M)^2 = B^2+2B.M+M^2 = \\ = B.(B+M)+B.M+M^2 = -(K_S+\sum_{i=1}^r C_i).B+B.M+M^2 \geq 0, \end{aligned}$$

since both $-(K_S+\sum_{i=1}^r C_i)$ and $M$ are nef. Hence, we have $M^2=0$ and $B.M=0$, which implies that $B^2=0$, since $(B+M)^2=0$.

We claim that $B$ is nef. Indeed, put $B=\sum_{i=1}^k a_iB_i$, where $B_i$ is an irreducible curve, and $a_i$ is a positive integers. Then

$$0 = (B+M).\Big(\sum_{i=1}^k a_iB_i\Big) \geq \sum_{i=1}^k a_i(B+M).B_i,$$

which implies that $(B+M).B_i = 0$ for every possible $i$. Similarly, we see that $M.B_i = 0$ for every possible $i$, which implies that $B.B_i = 0$ for every possible $i$. Hence, the divisor $B$ is nef.

Since $B$ is nef, we have $h^1(\mathcal{O}_S(B)) = h^2(\mathcal{O}_S(B)) = 0$ by the KawamataViehweg vanishing theorem. Applying the Riemann–Roch theorem to the divisor $B$, we see that

$$h^0(\mathcal{O}_S(B)) = 1 + \frac{B.(B-K_S)}{2} = 1 - \frac{B.K_S)}{2} \geq 0,$$

since $-K_S \sim \sum_{i=1}^r C_i + B + M$ and $B$ is nef. But $h^0(\mathcal{O}_S(B) = 1$, because $B$ is the fixed part of the linear system $|-(K_S + \sum_{i=1}^r C_i)|$. The latter implies that $-K_S.B = 0$. Since $B \neq 0$ by assumption, the Riemann–Roch theorem implies that

$$\begin{aligned} h^2(\mathcal{O}_S(-B)) = h^0(\mathcal{O}_S(-B)) + h^2(\mathcal{O}_S(-B)) = \\ = 1 + h^1(\mathcal{O}_S(-B)) + \frac{B.(B-K_S)}{2} = 1 + h^1(\mathcal{O}_S(-B)) \geq 1, \end{aligned}$$

which implies that $h^2(\mathcal{O}_S(-B)) \neq 0$. By Serre duality, we have $h^2(\mathcal{O}_S(-B)) = h^0(\mathcal{O}_S(B + K_S))$. But

$$B + K_S \sim -\sum_{i=1}^r C_i - M,$$

which implies that $h^0(\mathcal{O}_S(B + K_S)) = 0$, which is a contradiction. Thus, the linear system $|-(K_S + \sum_{i=1}^r C_i)|$ is free from fixed curves.

Since $|-(K_S + \sum_{i=1}^r C_i)|$ is free from fixed curves and $(-K_S - \sum_{i=1}^r C_i)^2 = 0$, the linear system $|-(K_S + \sum_{i=1}^r C_i)|$ does not have base points at all. $\square$

Since $(-K_S - \sum_{i=1}^r C_i)^2 = 0$, the linear system $|-(K_S + \sum_{i=1}^r C_i)|$ is composed from a base point free pencil. By Bertini theorem, there exists a smooth irreducible curve $F$ such that $F^2 = 0$, the linear system $|F|$ is a base point free pencil, and

$$-K_S - \sum_{i=1}^r C_i \sim kF$$

for some positive integer $k$. Since $-K_S$ is big, we have $-K_S.F > 0$. Hence, we have $-K_S.F = 2$ and $F \cong \mathbb{P}^1$ by adjunction formula. Then it follows from the Riemann–Roch theorem that $h^0(\mathcal{O}_S(kF)) = k + 1$, which implies that $k = l$.

We may assume that $F$ is a general curve in $|F|$. The pencil $|F|$ gives a morphism $\xi\colon S \to \mathbb{P}^1$ whose general fiber is $F \cong \mathbb{P}^1$, i.e., the morphism $\xi$ is a *conic bundle*. Since $-K_S.F = 2$ and $-(K_S + \sum_{i=1}^r C_i).F = 0$, we have $F.(\sum_{i=1}^r C_i) = 2$.

For every irreducible curve $Z$ on the surface that is contained in the fibers of $\xi$, we have

$$0 < -(K_S + \sum_{i=1}^r (1-\beta_i)C_i).Z \sim_{\mathbb{R}} F.Z + \sum_{i=1}^r \beta_i C_i.Z = \sum_{i=1}^r \beta_i C_i.Z,$$

for all $0 < |\beta| \ll 1$, because $(S, \sum_{i=1}^r C_i)$ is strongly asymptotically log del Pezzo (note that this step does not work if $(S, \sum_{i=1}^r C_i)$ just asymptotically log del Pezzo). This implies that $\sum_{i=1}^r C_i.Z > 0$. Keeping in mind that $\sum_{i=1}^r C_i.F = 2$, we see that either $\sum_{i=1}^r C_i.Z = 1$ or $\sum_{i=1}^r C_i.Z = 2$. In the latter case, we must have $Z \sim F$. This implies that $\xi$ is so-called *standard conic bundle*, i.e., every singular fiber of $\xi$ consists of a union of two smooth rational curves that intersect each other transversally at one point.

# 5 Reductivity of the automorphism group of a pair

Denote by $\mathrm{aut}(X)$ the Lie algebra of holomorphic vector fields on $(X, J)$, i.e., all vector fields $V \in \Gamma(X, TX)$ satisfying $L_V J = 0$. We emphasize that these are real vector fields. The projection of $V$ onto $T^{1,0}X$, denoted $V^{1,0} = (V - \sqrt{-1}JV)/2$ will be referred to as a holomorphic (1,0)-vector field, and it is sometimes convenient to work with $\mathrm{aut}(X)$ recast in this complex notation. Let $\mathrm{aut}(X, D) \subset \mathrm{aut}(X)$ denote the subspace of fields tangent to $D$. It is a Lie subalgebra.

**Proposition 5.1.** *Let $(X, D, \omega)$ be a KEE manifold. Then $\mathrm{aut}(X, D)$ is the complexification of the Lie algebra of (Hamiltonian) Killing vector fields of $(X, \omega)$.*

*Proof.* Suppose $V = \nabla u \in \mathrm{aut}(X, D)$ is a gradient holomorphic vector field (here $u$ is a real-valued function). We claim that $JV$ is a Killing field with respect to $g$ (the metric associated to $\omega$). Indeed, this is equivalent to $Z \mapsto \nabla_Z(JV) = J\nabla_Z V$ being a skew-symmetric endomorphism of $TX$ [38, Proposition 27]. But $\nabla(J\nabla u) = \nabla(\sqrt{-1}\nabla^{1,0}u - \sqrt{-1}\nabla^{0,1}u) = \sqrt{-1}(\nabla^{0,1}\nabla^{1,0}u - \nabla^{1,0}\nabla^{0,1}u) = -\sqrt{-1}\partial\bar{\partial}u$, since $\nabla^{1,0}V^{1,0} = \nabla^{1,0}\nabla^{1,0}u = 0$.

Next, we claim that any element $X$ of $\mathrm{aut}(X, D)$ is necessarily a linear combination of a gradient vector field and $J$ applied to such a field. In fact, consider the $(0, 1)$-form $g(V^{1,0}, \,.\,)$ given in local coordinates by $g_{i\bar{j}}V^i\overline{dz^j}$. Since $\nabla^{1,0}V = 0$, this form is closed. Thus, by Lemma 5.3 below, it equals a $\bar{\partial}$-exact form, say $\bar{\partial}u/2$ with $u$ complex-valued. It follows that $V^{1,0} = \nabla^{1,0}u/2$, and $V^{0,1} = \nabla^{0,1}\bar{u}/2$, so

$$V = \nabla \mathrm{Re}\, u + J\nabla \mathrm{Im}\, u. \tag{5.1}$$

The same argument also shows that any Killing field $V$ is necessarily a Hamiltonian vector field. In fact, an isometry homotopic to the identity preserves any $\omega$-harmonic form by Hodge theory [18, p. 82] since it preserves its class. Thus, $L_V\omega = 0$, or $\iota_V\omega = 0$. Since $b^1(V) = 0$ then $\iota_V\omega = du$ and $V = -J\nabla u$. Further, since $L_V g = 0$, $L_V\omega = 0$, and $\omega(\,.\,, \,.\,) = g(J\,.\,, \,.\,)$, then also $L_V J = 0$. Next, note that any automorphism of $(X, \omega)$ must preserve its singular set, i.e., $D$. Thus, combining all the above, $\mathrm{aut}(X, D)$ must contain the Lie algebra of Killing fields of $(X, \omega)$.

Thus, we would be done if we knew that each summand in the decomposition (5.1) of a holomorphic vector field $V$ were itself a holomorphic vector field (then $V$ would be equal to a Killing field and $J$ times such a field, by the previous paragraphs). To show that, recall that as shown in [24, §6], if $\omega$ is a KEE metric of positive Ricci curvature $\mu$, and $\phi$ is a complex-valued eigenfunction of $-\Delta_\omega$ with eigenvalue $\mu$ then $\nabla^{1,0}\phi$ is a holomorphic (1,0)-vector field tangent to $D$. We claim that the converse is true as well. Assuming this claim, there is an isomorphism between $\Lambda_\mu(-\Delta_\omega)$ (the aforementioned eigenspace) and $\mathrm{aut}(X, D)$ given by $u + \sqrt{-1}v \mapsto \nabla u + J\nabla v$, where $u, v \in C^\infty(X)$ are real-valued functions. By the remark at the beginning of this paragraph then, the proposition follows: indeed, for a complex-valued function in $\Lambda_\mu(-\Delta_\omega)$ it is immediate that both its real and imaginary parts are contained in $\Lambda_\mu(-\Delta_\omega)$ since $-\Delta_\omega$ is a linear operator.

Thus suppose that $u \in C^\infty(X \setminus D) \cap C^0(X)$ is such that $\nabla u \in \mathrm{aut}(X, D)$, so that $\nabla^{1,0}\nabla^{1,0}u = 0$ (the gradients here and below are with respect to the edge metric $\omega$ and the underlying complex structure). Thus, using the Weitzenböck formula [24, §6] and the KEE assumption,

$$\begin{aligned} \Delta_\omega|\nabla^{1,0}u|^2_\omega &= 2\mathrm{Ric}\,(\nabla^{1,0}u, \nabla^{0,1}u) + 2|\nabla^{1,0}\nabla^{1,0}u|^2 + 2(\Delta_\omega u)^2 + 4\omega(\nabla^{1,0}u, \nabla^{0,1}\Delta_\omega u) \\ &= 2\mu|\nabla^{1,0}u|^2_\omega + 2(\Delta_\omega u)^2 + 4\omega(\nabla^{1,0}u, \nabla^{0,1}\Delta_\omega u). \end{aligned} \tag{5.2}$$

To conclude then, it would suffice to integrate (5.2) and prove that

$$\int_X \Delta_\omega |\nabla^{1,0} u|^2_\omega \omega^n = 0, \qquad \int_X |\nabla^{1,0} u|^2_\omega \omega^n = -\int_X u \Delta_\omega u \omega^n, \tag{5.3}$$

and

$$-\int_X \omega(\nabla^{1,0} u, \nabla^{0,1} \Delta_\omega u)\omega^n = \int_X (\Delta_\omega u)^2 \omega^n. \tag{5.4}$$

Indeed, these identities then imply

$$\mu \int_X |\nabla^{1,0} u|^2_\omega \omega^n = \int_X (\Delta_\omega u)^2. \tag{5.5}$$

They also imply that

$$\begin{aligned}\int_X |\nabla^{1,0} u|^2_\omega \omega^n &= -\int_X u \Delta_\omega u \omega^n \leq ||u||_{L^2(X,\omega^n)} ||\Delta_\omega u||_{L^2(X,\omega^n)} \\ &\leq \mu^{-1/2} ||\Delta_\omega u||_{L^2(X,\omega^n)} ||\nabla^{1,0} u||_{L^2(X,\omega^n)}\end{aligned}$$

where we used the fact that since $\omega$ is KEE, the first positive eigenvalue of $-\Delta_\omega$ equals $\mu$ [24, Lemma 6.1]. Therefore, $\mu \int_X |\nabla^{1,0} u|^2_\omega \omega^n \leq \int_X (\Delta_\omega u)^2$, with equality if and only if $u$ is an eigenfunction of $-\Delta_\omega$ with eigenvalue $\lambda_1 = \mu$. Thus, by (5.5), $u$ is such an eigenfunction, concluding the proof of the proposition.

We now turn to proving (5.3)–(5.4). First, we claim that $u$ in fact has a polyhomogeneous expansion of the form

$$u \sim a_0(y) + (a_{10}(y)\cos\theta + a_{11}(y)\sin\theta) r^{\frac{1}{\beta}} + a_2(y) r^2 + O(r^{2+\eta}), \tag{5.6}$$

for some $\eta > 0$. Here $y$ is a local coordinate on $D$ and $re^{\sqrt{-1}\theta} = z_1^\beta$ with $D = \{z_1 = 0\}$ locally. For the proof, observe that since $\nabla^{1,0}\nabla^{1,0} u = 0$, $u$ lies in the kernel of the self-adjoint fourth-order Lichnerowicz operator $D_\omega := L^\star_\omega \circ L_\omega : C^\infty(X \setminus D) \cap C^0(X) \to C^\infty(X \setminus D) \cap C^0(X)$; here $L_\omega : u \mapsto \nabla^{1,0}\nabla^{1,0} u$ and $L^\star_\omega$ is the formal $L^2$ adjoint computed with respect to $\omega$. Second, $D_\omega$ is a linear degenerate elliptic operator of edge type, in the sense of Mazzeo [35], whose principal symbol is $\Delta^2_\omega$; more precisely [4, (2.1)],

$$D_\omega = \Delta^2_\omega + (\mathrm{Ric}\,\omega, \sqrt{-1}\partial\bar\partial(\,.\,))_\omega + (\partial s_\omega, \partial u)_\omega = \Delta^2_\omega + \mu\Delta_\omega, \tag{5.7}$$

by the KEE assumption. The expansion (5.6) then follows from the polyhomogeneous expansion for the KEE metric $\omega$ [24, Theorem 1] and the polyhomogeneous structure of inverses of elliptic edge operators associated to polyhomogeneous Kähler edge metrics [35, Theorem 6.1],[24, Proposition 3.8], and the fact that $u$ is bounded (if $u$ were not bounded then its expansion would contain a $\log r$ term, but then the corresponding vector field would not be bounded).

Finally, given (5.6), the verification of (5.3)–(5.4) follows in the same way as in the proof of [24, Lemma 6.1]. □

*Remark* 5.2. A shorter proof would be to avoid the Weitzenböck formula and and use (5.7) directly. It then follows that $v = \Delta_\omega u + \mu u$ is in the kernel of $\Delta_\omega$. By the asymptotic expansion for bounded solutions of $D_\omega u = 0$ we see that $v$ is bounded (indeed, the term of order $O(r^{1/\beta})$ in (5.6) is in the kernel of $\Delta_\omega$), and hence a constant. Then it follows that by changing $u$ by a constant it must be a eigenvalue of $\Delta_\omega$ with eigenvalue $-\mu$. We preferred the current proof since the Weitzenböck formula was used in [24] to obtain one direction of the isomorphism proved here, and it seemed natural to emphasize what is needed to make that proof work in the other direction.

**Lemma 5.3.** *Let $(X, D, \omega)$ be a Kähler edge manifold, and suppose that $c_1(X) - \sum_i(1 - \beta_i)[D_i] = \mu[\omega]$ with $\mu > 0$. Then $b_1(X) = 0$.*

*Proof.* This is a direct corollary of the Kawamata–Viehweg Vanishing Theorem which states that $H^i(X, \mathcal{O}_X(K_X + N)) = 0$ for all $i > 0$ whenever $N$ is numerically equivalent to a sum $B + \Delta$ of a big and nef $\mathbb{Q}$-divisor $B$, and a $\mathbb{Q}$-divisor with snc support $\Delta$ [31, Vol. II, §9.1.C]. Thus, we may choose $\beta \in \mathbb{Q}^N \cap (0,1)^N$ such that $B := -K_X - \sum_i(1 - \beta_i)D_i$ is ample, and set $N = \sum_i(1 - \beta_i)D_i$. Finally, by Hodge theory $b_1(X) = 2h^{1,0}(X) = 2\dim H^1(X, \mathcal{O}_X) = 0$ [18, p. 105]. □

*Remark* 5.4. In fact, it follows from [47, Corollary 1] that $X$ is simply connected in a much more general setting. This generalizes the classical result of Kobayashi in the Fano case [25].

Let $\mathrm{Aut}_0(X)$ denote the connected Lie group associated to $\mathrm{aut}(X)$. Similarly, denote by $\mathrm{Aut}_0(X, D) \subset \mathrm{Aut}_0(X)$ the Lie subgroup associated to $\mathrm{aut}(X, D)$. This is the identity component of the automorphism group of the pair. Putting the above results together we obtain a version of Matsushima's Theorem [34] for pairs.

*Proof of Theorem 1.12.* Suppose that $c_1(X) - (1 - \beta)[D] = \mu[\omega]$ with $\mu \in \mathbb{R}$. In case $\mu > 0$ the statement is a corollary of Propostion 5.1 since as noted in its proof every Killing vector field of $(X, g)$ is Hamiltonian.

Suppose now that $\mu \leq 0$. Let $\psi \in \mathrm{Aut}_0(X, D)$. Since $\psi \in \mathrm{Aut}(X)$, $\psi^\star c_1(X) = c_1(X)$. Since $\psi$ fixes $D$, $\psi^\star[D] = [D]$. Thus, $\psi^\star[\omega] = [\omega]$. Therefore, if $\omega$ is KEE then $\psi^\star\omega$ is a cohomologous KEE form. But, when $\mu \leq 0$ the KEE form is unique in its cohomology class [24, Theorem 2]. Thus $\psi$ is the identity map, and $\mathrm{Aut}_0(X, D) = \{\mathrm{id}\}$. □

*Remark* 5.5. Using the arguments above one can prove a corresponding generalization to the edge setting of Calabi's theorem on the structure of the automorphism group of an extremal metric [4]. For brevity, we do not go into the details here.

# 6 Tian invariants of asymptotic pairs

Throughout the article we use the standard language of the singularities of pairs [27, 6]. By strictly log canonical (lc) singularities we mean log canonical singularities that are not Kawamata log terminal [27, Definition 3.5]. We also distinguish between an $\alpha$-invariant as in Definition 6.3 below, by which we refer to a global log canonical threshold, and a Tian invariant, by which we refer to the analogous invariant defined analytically in terms of metrics [42]. These two invariants coincide under certain regularity assumptions [6, 1]. The algebraic definition makes sense in more general (singular and/or degenerate) settings, while the analytic definition is useful for proving existence of KEE metrics by Theorem 1.14.

## 6.1 A general bound on global log canonical thresholds of pairs

Given a proper birational morphism $\pi : Y \to X$, we define the exceptional set of $\pi$ to be the smallest subset $\mathrm{exc}(\pi) \subset Y$, such that $\pi : Y \setminus \mathrm{exc}(\pi) \to X \setminus \pi(\mathrm{exc}(\pi))$ is an isomorphism.

A log resolution of $(X, \Delta)$ is a proper birational morphism $\pi : Y \to X$ such that $\pi^{-1}(\Delta) \cup \{\mathrm{exc}(\pi)\}$ is divisor with snc support. Log resolutions exist for all the pairs we will consider in this article, by Hironaka's theorem.

Assume that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Given a log resolution of $(X, \Delta)$, write

$$\pi^\star(K_X + \Delta) = K_Y + \tilde{\Delta} + \sum e_i E_i,$$

where $\tilde{\Delta}$ denotes the proper transform of $\Delta$, and where $\mathrm{exc}(\pi) = \cup E_i$, and $E_i$ are irreducible codimension one subvarieties. Also, assume $\Delta = \sum \delta_i \Delta_i$, with $\Delta_i$ irreducible codimension one subvarieties, so $\tilde{\Delta} = \sum \delta_i \tilde{\Delta}_i$. Singularities of pairs can be measured as follows.

**Definition 6.1.** *Let $Z \subset X$ be a subvariety. A pair $(X, \Delta)$ has at most log canonical (lc) singularities along $Z$ if $e_i, \delta_j \leq 1$ for every $i$ such that $E_i \cap Z \neq \emptyset$ and every $j$ such that $\Delta_j \cap Z \neq \emptyset$.*

**Definition 6.2.** *Let $Z \subset X$ be a subvariety. The log canonical threshold of the pair $(X, \Delta)$ along $Z$ is*

$$\mathrm{lct}_Z(X, \Delta) := \sup\{\lambda : (X, \lambda\Delta) \text{ is log-canonical along } Z\}.$$

*Set* $\mathrm{lct}(X, \Delta) := \mathrm{lct}_X(X, \Delta)$.

Let $X$ be a variety, let $B$ and $D$ be effective Cartier $\mathbb{Q}$-divisors on the variety $X$ such that the singularities of the log pair $(X, B)$ are log terminal, and $K_X + B + D$ is a $\mathbb{Q}$-Cartier divisor. Recall that the log canonical threshold of the boundary $D$ is the number

$$\mathrm{lct}\big(X, B; D\big) = \sup\big\{\lambda \in \mathbb{Q} : \text{the pair } \big(X, B + \lambda D\big) \text{ is log canonical}\big\}.$$

Let $H$ be an ample $\mathbb{Q}$-divisor on $X$, and let $[H]$ be the class of the divisor $H$ in $\mathrm{Pic}(X) \otimes \mathbb{Q}$.

**Definition 6.3.** *The global log canonical threshold of the log pair $(X, B)$ with respect to $[H]$ is the number*

$$\alpha(X, B, [H]) := \inf\big\{\mathrm{lct}\big(X, B; D\big) : D \text{ is effective } \mathbb{Q}\text{-divisor such that } D \sim_{\mathbb{Q}} H\big\}.$$

For simplicity, we put $\alpha(X, [H]) = \alpha(X, B, [H])$ if there is no boundary, i.e., $B = 0$. Similarly, we put $\alpha(X, B) = \alpha(X, B, [H])$ if $H \sim_{\mathbb{Q}} -(K_X + B)$.

Finally, we put $\alpha(X) = \alpha(X, [H])$ if $B = 0$ and $H = -K_X$. Note that it follows from Definition 6.3 that

$$\alpha\big(X, B, [H]\big) = \sup\left\{ c \;\middle|\; \begin{array}{l} \text{for every } \mathbb{Q}\text{-divisor } D \text{ such that } D \sim_{\mathbb{Q}} H \\ \text{the log pair } (X, B + cD) \text{ is log canonical} \end{array} \right\}, \tag{6.1}$$

and $\alpha(X, B, [\mu H]) = \alpha(X, B, [H])/\mu$ for every positive rational number $\mu$.

By a result of Demailly [6, Appendix] (with complements by Berman [1] in the log setting) $\alpha(X, \sum(1 - \beta_i)D_i, [H])$ coincides with Tian's invariant for the Kähler class $[H]$ [42] when $X$ is smooth, $\sum D_i$ has simple normal crossings and when the background measure has edge singularities of angle $2\pi\beta_i$ along $D_i$. In other words

$$\alpha(X, \sum(1 - \beta_i)D_i, [H]) = \sup\Big\{ a : \sup_{\varphi \in \mathrm{PSH}(X, \omega_0)} \int_X e^{-a(\varphi - \sup\varphi)}\omega^n < \infty\Big\}, \tag{6.2}$$

where $\omega$ is a Kähler edge metric with angle $2\pi\beta_i$ along $D_i$ and $\omega_0$ is a smooth Kähler metric with $[\omega_0] = [\omega] = [H]$. In the notation of [24, §6.3] $\mu = 1$, so in this normalization the criterion for existence of KEE is precisely the one stated in Theorem 1.14.

The next lemma gives an explicit bound for $\alpha$-invariants on curves. We will make use of it in Proposition 6.10 to obtain explicit bounds for $\alpha$-invariants on log del Pezzo surfaces. It also serves to illustrate the definitions above.

**Lemma 6.4.** *Let $C$ be a smooth curve, $P_i \in C$ distinct points, and $a_i \geq 0$. Suppose that $(C, \sum_{i=1}^k a_i P_i)$ is log terminal, i.e., $a_i < 1$ for all $i$. Let $H$ be an ample $\mathbb{R}$-divisor on $C$, and let $d \in \mathbb{R}_{>0}$ be its degree. Then*

$$\alpha\Big(C, \sum_{i=1}^{k} a_i P_i, [H]\Big) \geq \frac{1 - \max\{a_1, \ldots, a_k\}}{d}.$$

*Furthermore, equality holds when $C = \mathbb{P}^1$.*

*Proof.* If $D \sim_{\mathbb{Q}} H$ then $D = \sum b_i Q_i$ with $b_i \geq 0$ and $\sum b_i = d$. Then $(C, \sum_{i=1}^n a_i P_i + \lambda D)$ is log canonical precisely when $a_i + \lambda b_i \leq 1$, i.e., $\lambda \leq (1 - a_i)/b_i$ for all such admissible $b_i$ (here we are assuming that $P_i = Q_i$ otherwise the bounds are even weaker). In particular, if $\lambda \leq (1 - \max_i a_i)/d$ then the pair is always log canonical. This proves the inequality. The result follows since we may choose a divisor $D = dP_j$ with $j$ such that $\max_i a_i = a_j$; on $\mathbb{P}^1$ $\mathbb{Q}$-rational equivalence is determined solely by degree so $D \sim_{\mathbb{Q}} H$, thus in this case $\alpha\Big(C, \sum_{i=1}^k a_i P_i, [H]\Big) \leq (1 - \max_i a_i)/d$. $\square$

The following gives a general bound on global log canonical thresholds of pairs.

**Proposition 6.5.** *Suppose that $B = (1 - \beta)S$, where $\beta \in (0, 1)$ and $S$ is an irreducible nef Cartier divisor on $X$. Let $H$ be an ample $\mathbb{Q}$-divisor on $X$. Put*

$$\gamma = \sup\big\{c \in \mathbb{Q} \bigm| H - cS \text{ is pseudoeffective}\big\}.$$

*Then $\alpha(X, (1 - \beta)S, [H]) \geq \min(\beta/\gamma, \alpha(X, [H]), \alpha(S, [H]|_S))$.*

*Proof.* Put $\lambda = \min(\beta/\gamma, \alpha(X, [H]), \alpha(S, [H]|_S))$. We may assume that $\lambda > 0$. Suppose that $\mathrm{lct}(X, (1 - \beta)S, [H]) < \lambda$. Then there exists an effective $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $\Delta \sim_{\mathbb{Q}} H$ and the log pair $(X, (1 - \beta)S + \mu\Delta)$ is not log canonical at some point $P \in X$ for some positive rational number $\mu < \lambda$.

If $P \notin S$, then the log pair $(X, \mu\Delta)$ is not log canonical at the point $P \in X$, contradicting $\mu < \lambda \leq \alpha(X, [H])$ and $\Delta \sim_{\mathbb{Q}} H$. Thus, $P \in S$.

Put $(1 - \beta)S + \mu\Delta = aS + R$ for some positive rational number $a \geq 1 - \beta$ and some effective $\mathbb{Q}$-divisor $R$ such that $S \not\subset \mathrm{Supp}(R)$. Since

$$H \sim_{\mathbb{Q}} \Delta = \frac{a - 1 + \beta}{\mu} S + \frac{1}{\mu} R,$$

we see that $(a - 1 + \beta)/\mu \leq \gamma$. Because $\mu < \lambda \leq \beta/\gamma$ then $a \leq 1$. Since $a \leq 1$, the log pair $(X, S + R)$ is not log canonical at the point $P \in S$. Thus, it follows from adjunction theorem [30, Theorem 5.50] that the log pair $(S, R|_S)$ is not log canonical as well.

Note that $R \sim_{\mathbb{Q}} \mu H - (a - 1 + \beta)S$. Thus, if $S|_S$ is $\mathbb{Q}$-linearly equivalent to some effective divisor $T_S$ on $S$, then

$$R|_S + (a - 1 + \beta)T_S \sim_{\mathbb{Q}} \mu H$$

while $(S, R|_S + (a - 1 + \beta)T_S)$ is not log canonical, which contradicts $\mu < \alpha(S, [H]|_S))$. Unfortunately, we do not know that $S|_S$ is $\mathbb{Q}$-linearly equivalent to some effective divisor on $S$, because we only know that $S|_S$ is nef. Nevertheless, we still can obtain a contradiction in a similar way by adding to $S|_S$ a small piece of an ample divisor $H|_S$. Note that

$$R \sim_{\mathbb{Q}} \mu H - (a - 1 + \beta)S = \lambda H - \Big((\lambda - \mu)H + (a - 1 + \beta)S\Big),$$

where $(\lambda-\mu)H+(a-1+\beta)S$ is an ample $\mathbb{Q}$-divisor, since $S$ is nef and $H$ is ample. Thus, there exists an effective $\mathbb{Q}$-divisor $G$ on the variety $X$ such that $G\sim_{\mathbb{Q}}(\lambda-\mu)H+(a-1+\beta)S$ and $S\not\subset\mathrm{Supp}(G)$. Then $(R+G)|_S\sim_{\mathbb{Q}}[\lambda H]|_S$ and the log pair $(S,(R+G)|_S)$ is not log canonical, since $(S,R|_S)$ is not log canonical and $G|_S$ is an effective $\mathbb{Q}$-divisor on $S$. On the other hand, the log pair $(S,(R+G)|_S)$ must be log canonical, because $\lambda\le\alpha(S,[H]|_S))$ and $\lambda^{-1}(R+G)|_S\sim_{\mathbb{Q}}[H]|_S$. $\square$

The previous result specializes to a result of Berman [1] when $X$ is further assumed to be smooth, when the boundary $S$ is assumed to be smooth and ample, and further when $S$ and $H$ are proportional in the sense that $S\sim_{\mathbb{Q}}cH$, (i.e., in his setting $S$ is a section of $H$ and $\gamma=c$). Upon completion of this article we learned that Odaka–Sun also gave an algebraic proof of Berman's result in the special case $[H]=[S]=-K_X$ [37, Corollary 5.5]. We decided to keep Proposition 6.5 due to its general form and possible application to polarizations different from $-K_X$. Thus, we obtain as a corollary the following result originally proved by Berman using analytic methods.

**Corollary 6.6.** *Suppose that $X$ is smooth, $B=(1-\beta)S$, where $\beta\in(0,1]$ and $S$ is an irreducible smooth ample Cartier divisor on $X$. Then*

$$\alpha\big(X,(1-\beta)S,[\beta S]\big)\ge\min\left\{1,\frac{\alpha(X,[S])}{\beta},\frac{\alpha(S,[S]|_S)}{\beta}\right\}.$$

## 6.2 Limiting behavior of $\alpha$-invariants

In this subsection we prove Theorem 1.5. The proof is divided into Propositions 6.8, 6.9, and 6.10.

More generally, we conjectured in (1.1) that in higher dimensions

$$\lim_{\beta\to0^+}\alpha(X,(1-\beta)D)=\begin{cases}1 & \text{if } K_X+D\sim0,\\ \min\{1,\alpha(X,[-K_X-D]),\alpha(D)\} & \text{if } 0\not\sim-K_X-D \text{ is not big},\\ 0 & \text{if } -K_X-D \text{ is big.}\end{cases}$$

provided that $D$ is irreducible and smooth. Note that the divisors $-K_X-D$ and $-K_S=(-K_X-D)|_D$ may not be ample. This violates our definitions of $\alpha(X,[-K_X-D])$ and $\alpha(D)=\alpha(D,[(-K_X-D)|_D])$. However, it follows from [30, Theorem 3.3] that $-K_X-D$ and $-K_D$ are semiample, so we can define $\alpha(X,[-K_X-D])$ and $\alpha(D)$ in the same way as in the case when $-K_X-D$ and $-K_D$ are ample (and it could happen that $\alpha=\infty$, for instance). We prove this conjecture in the cases when $K_X+D\sim0$ or $-K_X-D$ is big. While Proposition 6.9 is two-dimensional, we believe its proof should find a suitable generalization to higher dimensions.

*Remark* 6.7. The situation in the simple normal crossings case is more complicated and there is in general no unique limit for Tian's invariant as $|\beta|$ tends to zero. To illustrate this we consider the following toric example. Let $L_1$, $L_2$, $L_3$ be distinct lines on $\mathbb{P}^2$. Then

$$\alpha(\mathbb{P}^2,\sum_{i=1}^3(1-\beta_i)L_i)=\frac{\max(\beta_1,\beta_2,\beta_3)}{\beta_1+\beta_2+\beta_3}$$

for any $(\beta_1,\beta_2,\beta_3)\in(0,1]^3$. Furthermore, let $G$ be a finite group in $\mathrm{Aut}(\mathbb{P}^2)$ such that $L_1+L_2+L_3$ is $G$-invariant and $G$ does not fix any point in $\mathbb{P}^2$ (there are infinitely many such groups). Then $\alpha_G(\mathbb{P}^2,(1-\beta)\sum_{i=1}^3L_i)=1$ for any $\beta\in(0,1]$. The proof, modelled on the arguments in [6, Lemma 5.1.], is left to the reader.

### 6.2.1 Class ($\gimel$) and ($\daleth$)

The next result holds for asymptotically log Fano varieties in any dimension and for $C$ with any $r \geq 1$.

**Proposition 6.8.** *Suppose that $-K_S - C$ is big. Then $\lim_{\beta \to 0^+} \alpha(S, (1-\beta)C) = 0$.*

*Proof.* Since $-K_S - C$ is big, there exists positive integer $N$ such that

$$-N(K_S + C) \sim C + \Delta$$

for some effective divisor $\Delta$ (this follows from the characterization of bigness [31, Corollary 2.2.7] since given an ample class $H$ and an effective class $C$ one may find an integer $M$ such that $MH \sim C + E$ for some effective divisor $E$). Put $\epsilon = \frac{1}{N}$. Then $-K_S - C \sim_{\mathbb{Q}} \epsilon C + \epsilon\Delta$. Take any sufficiently small real $\beta > 0$ such that $-(K_S + (1-\beta)C)$ is ample. Put $D = (\beta+\epsilon)C + \epsilon\Delta$. Then

$$D \sim_{\mathbb{R}} (\beta + \epsilon)C + \epsilon\Delta \sim_{\mathbb{R}} -(K_S + (1-\beta)C).$$

On the other side, we have $\mathrm{lct}(S, (1-\beta)C; D) \leq \frac{\beta}{\epsilon+\beta}$, which implies that $\alpha(S, (1-\beta)C) \leq \frac{\beta}{\epsilon+\beta}$. This shows that

$$\lim_{\beta \to 0^+} \alpha(S, (1-\beta)C) = 0,$$

since $\epsilon$ depends only on the pair $(S, C)$ and not on $\beta$. $\square$

### 6.2.2 Class ($\beth$)

In this subsection we prove Theorem 1.5 in the case when $(S, C)$ is of class ($\beth$).

Before embarking on the proof let us say few words about the idea of the proof. If $-K_S - C$ is not big, then by Proposition 1.7 $C \cong \mathbb{P}^1$ and $|-K_S - C|$ is free from base points and gives a morphism $S \to \mathbb{P}^1$ whose general fiber is $\mathbb{P}^1$ (a conic bundle). Moreover, the general curve in $|-K_S - C|$ is a fiber of this conic bundle. On the other hand, when $\beta$ is small, the class

$$-K_S - (1-\beta)C \sim_{\mathbb{R}} -(K_S + C) + \beta C$$

is close to $-K_S - C$. Thus, when looking for divisors

$$\Delta \sim_{\mathbb{R}} -(K_S + (1-\beta)C)$$

having small $\mathrm{lct}(S, (1-\beta)C; \Delta)$ with $0 < \beta \ll 1$, there are not many options. Namely, we can take $\Delta$ to be $\beta C + F$ where $F$ is a fiber of the conic bundle. All other choices of $\Delta$ gives us either better or similar singularities. The reason is a *continuity* of $\alpha(S, (1-\beta)C)$ in $\beta$. When $\beta$ is very small, we have

$$\alpha(S, (1-\beta)C) \approx \alpha(S, C).$$

Note that $\alpha(S, C)$ is not well defined according to our definition of the $\alpha$-invariant, because $-K_S - C$ is not ample. Nevertheless, we can still define $\alpha(S, C)$ in a similar way, since $-(K_S + C)$ is semi-ample. On the other hand, if $\beta = 0$, we have no freedom in choosing $\Delta$ at all! Indeed, if $\beta = 0$, then

$$\Delta \sim_{\mathbb{R}} -K_S - C,$$

which implies that every irreducible component of $\Delta$ must be a fiber of the conic bundle $S \to \mathbb{P}^1$. In this case, the worst $\Delta$ (the one with smallest $\mathrm{lct}(S, C; \Delta)$) must be a fiber of the conic bundle. Furthermore, among these fibers there are exactly two that are worse than others, i.e., the two fibers that do not intersect $C$ transversally. So in a sense we have a choice choice of exactly two divisors for $\Delta$, which both gives us $\mathrm{lct}(S, C; \Delta) \approx \frac{1}{2}$.

**Proposition 6.9.** *Suppose that $-K_S - C$ is not big. Then* $\lim_{\beta\to 0^+}\alpha(S,(1-\beta)C) = \frac{1}{2}$.

*Proof.* By Lemma 2.2, we have $C \cong \mathbb{P}^1$. By Proposition 1.7 the linear system $|-K_S - \sum_{i=1}^r C_i|$ is free from base points and gives a morphism $\xi\colon S \to \mathbb{P}^1$ such that its general fiber is $\mathbb{P}^1$, and every reducible fiber consists of exactly two components.

Let $F$ be a general fiber of $\xi$. Then $-K_S - C \sim F$, since $|-K_S - \sum_{i=1}^r C_i|$ is a pencil by Lemmas 4.3 and 4.4. Then $F.C = 2$, since $-K_S - C^2 = 0$.

The morphism $\xi$ induces a double cover $C \to \mathbb{P}^1$. Since $C$ is a smooth rational curve, this double cover has exactly two ramification points. Let $O$ be one of these two ramification points, and let $F_O$ be a fiber of $\xi$ that passes through it. Recall that

$$-(K_S + (1-\beta)C) \sim_{\mathbb{R}} F_O + \beta C$$

by construction. On the other hand, we have

$$\mathrm{lct}(S,(1-\beta)C; F_O + \beta C) = \begin{cases} \dfrac{1+\beta}{2+\beta} & \text{if } F_O \text{ is singular,} \\ \dfrac{1+2\beta}{2+2\beta} & \text{if } F_O \text{ is smooth.} \end{cases}$$

To see this it suffices to blow-up once when $F_O$ is singular and twice when it is smooth. Hence, $\alpha(S,(1-\beta)C) \leq (1+\beta)/(2+\beta)$. To complete the proof it is thus enough to show that for every positive real $\epsilon > 0$ there exists real $\delta = \delta(\epsilon, C) > 0$ such that both (2.1) and

$$\alpha(S,(1-\beta)C) \geq \frac{1}{2} - \epsilon \tag{6.3}$$

for every real $\beta \in (0,\delta)$. In fact, we claim that $\delta = \min\{1/2, \epsilon/|C^2|, \beta_{\max}\}$ will do, where (2.1) holds for $\beta \in (0, \beta_{\max})$.

To that end we work with the definition (6.2) of the global log canonical threshold of the pair $(S,(1-\beta)C)$. We use repeatedly the following application of adjunction: if $K \subset S$ is a smooth irreducible curve and $M$ an effective $\mathbb{R}$-divisor on $S$ and if $(S, K+M)$ is not lc at a point $Q$ on $K$ then $(K, M|_K)$ is not lc at $Q$, or equivalently $\mathrm{mult}_Q K.M > 1$ [7, Excercise 6.31].

Throughout the proof we let $D$ be an effective $\mathbb{R}$-divisor satisfying

$$D \sim_{\mathbb{R}} F + \beta C$$

If the pair $(S,(1-\beta)C + \lambda D)$ is not lc at some point $P \in C$ and $C \not\subset \mathrm{Supp}(D)$ then

$$2 + \beta C^2 = C.\big(F + \beta C\big) = C.D \geq \mathrm{mult}_P\big(C.D\big) > \lambda^{-1},$$

thus $\lambda > \frac{1}{2+\beta C^2}$. If $(S,(1-\beta)C + \lambda D)$ is not lc at some point $P \in C$ and $C \subset \mathrm{Supp}(D)$ then write $D = \mu C + \Omega$, where $\mu$ is a positive rational number, and $\Omega$ is an effective $\mathbb{R}$-divisor on the surface $S$ whose support does not contain the curve $C$. Then

$$2\beta = (F + \beta C).F = D.F = (\mu C + \Omega).F = 2\mu + \Omega.F \geq 2\mu,$$

so $\mu \leq \beta$. On the other hand, $(S,(1-\beta+\lambda\mu)C + \lambda\Omega)$ is not lc at $P$. Since $1-\beta+\lambda\mu \leq 1$, also $(S, C + \lambda\Omega)$ is not lc at $P$. Thus,

$$2 + (\beta-\mu)C^2 = C.\big(F + (\beta-\mu)C\big) = C.\Omega \geq \mathrm{mult}_P\big(C.\Omega\big) > \lambda^{-1},$$

so again $\lambda > \frac{1}{2+\beta C^2}$.

Next, suppose that $(S,(1-\beta)C+\lambda D)$ is not lc at some point $P\notin C$. Then $(S,\lambda D)$ is not lc at $P$. Let $F_P$ be the fiber of $\xi$ that passes through $P$. Then we must consider three cases: $F_P$ is smooth, $F_P$ is singular and $P\neq \mathrm{Sing}(F_P)$, $F_P$ is singular and $P=\mathrm{Sing}(F_P)$.

First, suppose $F_P$ is smooth and put $D=\tau F_P+\Delta$, where $0<\tau\in\mathbb{Q}$, and $\Delta$ is an effective $\mathbb{R}$-divisor with $F_P\not\subset \mathrm{Supp}(\Delta)$. Then

$$4\beta+\beta^2C^2=D^2=(F_P+\beta C).D=(\tau F_P+\Delta).D=2\beta\tau+\Delta.D\geq 2\beta\tau,$$

so $\tau\leq 2+\frac{\beta}{2}C^2$. If $\lambda\tau>1$ then $\lambda>\frac{1}{2+\beta C^2/2}$. Suppose that $\lambda\tau\leq 1$. Thus, the pair $(S,F_P+\lambda\Delta)$ is not lc at $P$. Then

$$2\beta=F_P.\big(F_P+\beta C-\tau F_P\big)=F_P.\Delta\geq \mathrm{mult}_P\big(F_P.\Delta\big)>\lambda^{-1},$$

so $\lambda>\frac{1}{2\beta}$.

Next, suppose $F_P$ is singular. Then $F_P=F_1+F_2$, where $F_1$ and $F_2$ are smooth rational curves on $S$ such that $F_1.F_2=F_1.C=1=F_2.C=1$, and $F_1^2=F_2^2=-1$. Put $D=\tau_1F_1+\tau_2F_2+\Theta$, where $0<\tau_1,\tau_2\in\mathbb{Q}$, and $\Theta$ is an effective $\mathbb{R}$-divisor with $F_1,F_2\not\subset\mathrm{Supp}(\Theta)$. Then

$$\beta=(F+\beta C).F_1=(\tau_1F_1+\tau_2F_2+\Theta).F_1=-\tau_1+\tau_2+\Theta.F_1\geq -\tau_1+\tau_2, \qquad (6.4)$$

and similarly $\tau_1-\tau_2\leq\beta$. On the other hand, using that $D$ is ample we have

$$4\beta+\beta^2C^2=(F+\beta C).D=(\tau_1F_1+\tau_2F_2+\Theta).D=\beta(\tau_1+\tau_2)+\Theta.D\geq\beta(\tau_1+\tau_2),$$

so $\tau_1+\tau_2\leq 4+\beta C^2$, and combined with (6.4) then $\tau_2\leq 2+\frac{\beta}{2}C^2+\frac{\beta}{2}$ and similarly for $\tau_1$. If $\lambda\tau_i>1$ for some $i$ then $\lambda>\frac{1}{2+\beta C^2/2+\beta/2}$. Suppose that $\lambda\tau_1,\lambda\tau_2\leq 1$. There are two cases to consider: $P=F_1\cap F_2$ and $P\neq F_1\cap F_2$. Suppose first that $P=F_1\cap F_2$. Then the pairs $(S,F_1+\lambda\tau_2F_2+\lambda\Delta)$ and $(S,\lambda\tau_1F_1+F_2+\lambda\Theta)$ are not lc at $P$. Then

$$\beta+\tau_1=F_1.(F+\beta C-\tau_1F_1)=F_1.(\tau_2F_2+\Theta)>\tau_2+\lambda^{-1}$$

so $\lambda>\frac{1}{\beta+\tau_1-\tau_2}$ and similarly $\lambda>\frac{1}{\beta+\tau_2-\tau_1}$, so using (6.4) $\lambda>\frac{1}{2\beta}$. Next, suppose $P\neq F_1\cap F_2$, say $P\notin F_1$, $P\in F_2$. Hence, the log pair $(S,\lambda\tau_2F_2+\lambda\Theta)$ is not lc at $P$. Then

$$\beta+\tau_2-\tau_1=F_2.\big(F+\beta C-\tau_1F_1-\tau_2F_2\big)=F_2.\Theta>\lambda^{-1},$$

so $\lambda>\frac{1}{\beta+\tau_2-\tau_1}$, so again using (6.4) $\lambda>\frac{1}{2\beta}$

In conclusion, we see that if $(S,(1-\beta)C+\lambda D)$ is not lc then $\lambda>\frac{1}{2}-\epsilon$ whenever $\beta<\min\{1/2,\epsilon/|C^2|,\beta_{\max}\}$. Thus, (6.3) follows from (6.2), concluding the proof. □

### 6.2.3 Class (ℵ)

By Corollary 6.6, $\alpha(X,(1-\beta)S)\geq\min\{1,\beta^{-1}\alpha(X),\beta^{-1}\alpha(S,[S]|_S)\}$ when $(S,C)$ is of class (ℵ). Moreover, from the definition and the fact that $D\sim -K_X$ this invariant is bounded above by 1. Combining this, Lemma 6.4, and Theorem 1.14 yields:

**Proposition 6.10.** *Let $(X,D)$ be an asymptotically log Fano pair with $D\in|-K_X|$ a smooth irreducible divisor and $X$ Fano. Then $\lim_{\beta\to 0^+}\alpha(S,(1-\beta)C)=1$. Moreover,*
*(i) if $\dim X=2$, then $\alpha(X,(1-\beta)D)\in[\min\{1,\frac{1}{9\beta}\},1]$, and $(X,D)$ admits KEE metrics for all $\beta\in(0,1/6)$.*
*(ii) if $\dim X=3$, then $\alpha(X,(1-\beta)D)\in[\min\{1,\frac{1}{64\beta}\},1]$, and $(X,D)$ admits KEE metrics for all $\beta\in(0,1/48)$.*

*Proof.* As noted above, it suffices to estimate $\min\{\alpha(X), \alpha(D, [D]|_D)\}$.

(i) First, by using Lemma 6.4 and the fact that $K_X^2 \leq 9$ for every smooth del Pezzo surface (by their classification) one has $\alpha(D, [D]|_D) \geq 1/9$.

It remains to show that $\alpha(X) \geq 1/9$. This follows from the complete list of lcts of del Pezzo surfaces [5, Theorem 1.7] but we now explain a direct derivation that can also be adapted to prove (ii). Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $\Delta \sim_{\mathbb{Q}} -K_X$ and the log pair $(X, \lambda\Delta)$ is not log canonical at some point $P \in X$ for some positive rational $\lambda$.

If $-K_X$ is very ample, let $H_P$ be a general curve in $|-K_X|$ that passes through $P$. By the very ampleness we may assume that $H_P$ is not contained in the support of the divisor $D$. Thus, as in the proof of Proposition 6.9,

$$K_X^2 = \Delta.H_P \geq \mathrm{mult}_P(\Delta)\mathrm{mult}_P(H_P) \geq \mathrm{mult}_P(\Delta) > \lambda^{-1},$$

so $\lambda > 1/9$.

If $-K_X$ is not very ample but still base-point free $|-K_X|$ gives a surjective finite morphism $X \to V$, where $V$ is a surface, which implies that we can still proceed as in the very ample case. If $-K_X$ is not base-point free by the classification of del Pezzo surfaces $K_X^2 = 1$ and the linear system $|-2K_X|$ is base-point free and gives a surjective finite morphism $X \to V'$, where $V'$ is a surface. Let $H_P'$ denote a general curve in $|-2K_X|$ that passes though $P$. Then

$$2K_X^2 = \Delta.H_P \geq \mathrm{mult}_P(\Delta)\mathrm{mult}_P(H_P) \geq \mathrm{mult}_P(\Delta) > \lambda^{-1},$$

so $\lambda > 1/2K_X^2 = 1/2$. The result now follows from (6.1).

(ii) Let $\Delta$ be as in (i). Suppose first that $|-K_X|$ is base-point free. We claim that $\mathrm{mult}_P(\Delta) \leq 64$ for every point $P \in X$ and for every divisor $\Delta$ on $X$ such that $\Delta \sim_{\mathbb{Q}} -K_X$. Indeed, since $|-K_X|$ is base-point free, the linear system $|-K_X|$ gives a finite surjective morphism $X \to U$, where $U$ is a threefold. Thus, there exists $S_P$ and $S_P'$ in $|-K_X|$ such that $P \in \mathrm{Supp}(S_P.S_P')$ and no component of the 1-cycle $S_P.S_P'$ is contained in the support of $D$. Then

$$-K_X^3 = \Delta.S_P.S_P' \geq \mathrm{mult}_P(\Delta)\mathrm{mult}_P(S_P)\mathrm{mult}_P(S_P') \geq \mathrm{mult}_P(\Delta) > \lambda^{-1}.$$

Thus, $\lambda > -1/K_X^3 = 1/64$ [23, Corollary 7.1.2], implying $\alpha(X) \geq 1/64$. Similarly, we can prove that $\mathrm{mult}_P(\Omega) \leq 64$ for every point $P \in D$ and for every divisor $\Omega$ on $D$ such that $\Omega \sim_{\mathbb{Q}} -K_X|_D$. Thus, $\alpha(D, [D]|_D) \geq 1/64$.

Next, suppose that $|-K_X|$ has base-points. This is a very special situation. Indeed, it follows from [23, Theorem 2.4.5] that either $-K_X^3 = 4$ and $X$ is a blow up of a smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6 along a smooth elliptic curve that is a complete intersection of two surfaces in $|-\frac{1}{2}K_X|$, or $-K_X^3 = 6$ and $X \cong \mathbb{P}^1 \times S_1$, where $S_1$ is a smooth del Pezzo surface with $K_{S_1}^2 = 1$. In both cases $|-2K_X|$ is base-point free. Thus, the same arguments as in the base-point free case show that $\mathrm{mult}_P(\Delta) < -4K_X^3$ for every point $P \in X$ and for every divisor $\Delta$ on $X$ such that $\Delta \sim_{\mathbb{Q}} -K_X$, and that $\mathrm{mult}_P(\Omega) < -4K_X^3$ for every point $P \in D$ and for every divisor $\Omega$ on $D$ such that $\Omega \sim_{\mathbb{Q}} -K_X|_D$. Keeping in mind that $-K_X^3 \leq 6$, we see that $\alpha(X) \geq 1/24$ and $\alpha(D, [D]|_D) \geq 1/24$. □

Note that the lower bounds in Proposition 6.10 can be improved by a case-by-case analysis using results from [5, 6]. When $\dim X = 2$ it is also possible to say more about the existence of KEE metrics. In fact, in the cases (I.1A) and (I.5A.m) with $m \geq 3$ a KEE metric exists for all $\beta \in (0, 1]$ since it exists for $\beta = 1$ [1, 32, 24, 43]. In the remaining two cases (I.5A.m), $m \in \{1, 2\}$, it is possible to compute $\alpha(S, (1-\beta)C)$ to find all $\beta \in (0, 1]$ such that $\alpha(S, (1-\beta)C) > \frac{2}{3}$. Moreover, in the latter cases the value of the $\alpha$-invariant depends on the choice of the anticanonical boundary curve itself.

*Remark* 6.11. Let $X$ be a smooth Fano variety of dimension $n$, and let $D$ be a smooth divisor in $|-K_X|$. Put $M = 3^n(2^n-1)^n(n+1)^{n(n+2)(2^n-1)}$ and $N = 2(n+1)(n+2)!$. Then

$$\alpha(X,(1-\beta)D) \geq \min\{1, \beta^{-1}N^{n-1}M\},$$

for every $\beta \in (0,1]$. Indeed, $(-K_X)^n \leq 3^n(2^n-1)^n(n+1)^{n(n+2)(2^n-1)}$ (see, e.g., [8, Theorem 5.18]). On the other hand, $|-NK_X|$ is base-point free by [26, Theorem 1]. Note that $-12n^nK_X$ is very ample by [10, Corollary 12.11]. Thus we can proceed as in the proof of Proposition 6.10.

# 7 Existence and non-existence of KEE metrics

Our goal in this section is to make several first steps towards the uniformization of asymptotically log del Pezzo surfaces as stated in Conjecture 1.6.

## 7.1 Automorphism groups

Theorem 1.13 is a direct consequence of Theorem 1.12 and the following result.

**Proposition 7.1.** *The automorphisms groups of the following pairs of class* ($\gimel$) *or* ($\daleth$) *are not reductive:* (I.1C)*,* (I.2.n) *with any* $n \geq 0$*,* (I.6C.m) *with any* $m \geq 1$*,* (I.7.n.m) *with any* $n \geq 0$ *and* $m \geq 1$*,* (I.6B.1)*,* (I.8B.1) *and* (I.9C.1)*. On the other hand,* $\mathrm{Aut}(S,C)$ *is reductive when* $(S,C)$ *is one of the following:* (I.1A)*,* (I.4A)*,* (I.3B)*,* (I.4C)*,* (I.5.m) *with* $m \geq 1$*,* (I.1B)*,* (I.6B.m)*,* (I.8B.m)*, or* (I.9C.m) *with* $m \geq 2$*.*

*Proof.* If $(S,C)$ is (I.1A), (I.4A), or (I.5.m), then $\mathrm{Aut}(S,C)$ is finite, since $C$ is a $\mathrm{Aut}(S,C)$-invariant elliptic curve that is an anticanonical ample divisor. If $(S,C)$ is (I.1B) then $\mathrm{Aut}(S,C) \cong \mathrm{PGL}_2(\mathbb{C})$. If $(S,C)$ is (I.3B) then $\mathrm{Aut}(S,C) \cong \mathrm{GL}_2(\mathbb{C})$. If $(S,C)$ is (I.4C) then $\mathrm{Aut}_0(S,C) \cong \mathrm{PGL}_2(\mathbb{C})$.

For the case (I.1C), or, in fact, in any dimension, the pair $(\mathbb{P}^n, H)$ with $H$ a hyperplane in $\mathbb{P}^n$, satisfies

$$\mathrm{Aut}(\mathbb{P}^n, H) \cong \mathrm{Aut}(\mathbb{P}^n, p) \cong \mathrm{Aut}(\mathrm{Bl}_p\mathbb{P}^n) \cong \mathbb{G}_a^n \rtimes \mathrm{GL}_n(\mathbb{C}),$$

for a point $p \in \mathbb{P}^n$, where $\mathrm{Bl}_p\mathbb{P}^n$ denotes the blow-up of $\mathbb{P}^n$ at $p$. The latter group is not reductive. Note that this generalizes Troyanov's obstruction to the existence of a constant curvature metric on the teardrop ($S^2$ with one cone point).

In the case (I.2.0), we have

$$\mathrm{Aut}(\mathbb{F}_n, Z_n) \cong \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{Aut}(\mathbb{C}^1) \cong \mathrm{PGL}_2(\mathbb{C}) \times (\mathbb{G}_a \rtimes \mathbb{G}_m),$$

which is not reductive. In the case (I.2.n) with $n \geq 1$, we have $\mathrm{Aut}(\mathbb{F}_n, Z_n) \cong \mathrm{Aut}(\mathbb{F}_n)$, because the curve $Z_n$ must be fixed by any automorphism of $\mathbb{F}_n$ (since $n > 0$). On the other hand, it follows from [13, Theorem 4.10] that if $n > 0$, then

$$\mathrm{Aut}(\mathbb{F}_n) \cong \mathbb{G}_a^{n+1} \rtimes (\mathrm{GL}_2(\mathbb{C})/\mu_n),$$

where $\mathrm{GL}_2(\mathbb{C})/\mu_n$ acts on $\mathbb{G}_a^{n+1}$ by means of its natural linear representation in the space of binary forms of degree $n$. The latter group is not reductive.

If $(S,C)$ is in (I.6C.1), then $\mathrm{Aut}(S,C) \cong \mathbb{G}_a^2 \rtimes (\mathbb{G}_a \rtimes \mathbb{G}_m^2)$. If $(S,C)$ is in (I.6C.2), then $\mathrm{Aut}_0(S,C) \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$. If $(S,C)$ is in (I.6C.m) with $m \geq 3$, then $\mathrm{Aut}_0(S,C) \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_m$. All these groups are not reductive.

Now let us consider the case (I.7.n.m) for $m > 0$. If $(S, C)$ is in (I.7.0.1), then $\mathrm{Aut}(S, C) \cong (\mathbb{G}_a \rtimes \mathbb{G}_m) \times (\mathbb{G}_a \rtimes \mathbb{G}_m)$. If $(S, C)$ is in (I.7.0.2), then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m \times (\mathbb{G}_a \rtimes \mathbb{G}_m)$. If $(S, C)$ is in (I.7.0.m) with $m \geq 3$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a \rtimes \mathbb{G}_m$. If $(S, C)$ is in (I.7.1.1), then $\mathrm{Aut}(S, C) \cong \mathbb{G}_a^2 \rtimes (\mathbb{G}_a \rtimes \mathbb{G}_m^2)$. If $(S, C)$ is in (I.7.1.2), then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$. If $(S, C)$ is in (I.7.1.m) with $m \geq 3$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a^2 \rtimes \mathbb{G}_m$.

If $(S, C)$ is in (I.7.n.1) with $n \geq 2$, then

$$\mathrm{Aut}(S, C) \cong \mathbb{G}_a^{n+1} \rtimes ((\mathbb{G}_a \rtimes \mathbb{G}_m^2)/\mu_n),$$

where $((\mathbb{G}_a \rtimes \mathbb{G}_m^2)/\mu_n \subset \mathrm{GL}_2(\mathbb{C})/\mu_n$ acts on $\mathbb{G}_a^{n+1}$ by means of its natural linear representation in the space of binary forms of degree $n$. Similarly, if $(S, C)$ is in (I.7.n.2) with $n \geq 2$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a^{n+1} \rtimes \mathbb{G}_m^2/\mu_n$. Finally, if $(S, C)$ is in (I.7.n.m) with $n \geq 2$ and $m \geq 3$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a^{n+1} \rtimes \mathbb{G}_m/\mu_n$. All these groups are not reductive.

Now let us consider the case (I.6B.m) with $m \geq 1$. If $(S, C)$ is in (I.6B.1), then $\mathrm{Aut}(S, C) \cong \mathbb{G}_a \rtimes \mathbb{G}_m^2$, which is not reductive group. If $m = 2$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m$, which is reductive. If $m \geq 3$, then $\mathrm{Aut}(S, C)$ is finite.

Now let us consider the case (I.8B.m) with $m \geq 1$. If $m = 1$, then $\mathrm{Aut}(S, C) \cong \mathbb{G}_a \rtimes \mathbb{G}_m^2$, which is not reductive group. If $m = 2$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m^2$, which is reductive. If $m \geq 3$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m$, which is reductive.

Finally let us consider the case (I.9C.m) with $m \geq 1$. If $(S, C)$ is in (I.9C.1), then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_a \rtimes \mathbb{G}_m$, which is not reductive group. If $m = 2$, then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m$, which is reductive. If $m \geq 3$, then $\mathrm{Aut}(S, C)$ is finite. □

The following result shows that all pairs of class (ℶ) have reductive automorphism groups. This gives further evidence for Conjecture 1.6.

**Theorem 7.2.** *Let $(S, C)$ be a pair of class (ℶ) with $C$ smooth and irreducible. Then* $\mathrm{Aut}(S)$ *is reductive.*

*Proof.* If $(S, C)$ is (I.3A), then $\mathrm{Aut}_0(S, C) \cong \mathbb{G}_m$ (this is easy). If $(S, C)$ is (I.4B), then $\mathrm{Aut}_0(S, C)$ is a subgroup in $\mathrm{PGL}_2(\mathbb{C})$ that fixes two points (the ramification points of the double cover projection $C \to \mathbb{P}^1$), which implies that $\mathrm{Aut}_0(S, C)$ is either trivial or $\mathbb{G}_m$. Thus, if $(S, C)$ is (I.3A) or (I.4B) then $\mathrm{Aut}(S)$ is reductive. Note that this also follows from Theorem 1.12 combined with Theorem 1.15.

Suppose $(S, C)$ is (I.9B.m) with $m \geq 1$. Then $\mathrm{Aut}_0(S, C)$ preserves the conic bundle given by $|-K_S - C|$ (see Proposition 1.7). This implies that $\mathrm{Aut}_0(S, C)$ is a subgroup of the group $\mathrm{Aut}_0(S', C')$ where $(S', C')$ is a minimal model (see the proof of Theorem 2.1) of $(S, C)$ (it is either (I.3A) or (I.4B)). Thus, we see that $\mathrm{Aut}_0(S, C)$ is a subgroup of $\mathbb{G}_m$, which is either trivial of $\mathbb{G}_m$. In particular, we see that $\mathrm{Aut}_0(S, C)$ is reductive. □

## 7.2 Existence of KEE metrics on some pairs of class ℶ

The goal of this subsection is to prove Theorem 1.15 as a first step towards confirming Conjecture 1.6. This gives the first examples of pairs with KEE metrics of positive Ricci curvature which are not of class (ℵ). In §7.2.1 we define the $G$-invariant Tian invariant, with $G$ a finite group of automorphisms. In the remainder of this subsection we then compute the Tian invariants of three pairs of class ℶ. For the first two (I.3A), (I.4B) the surface is fixed ($\mathbb{F}_1$ or $\mathbb{P}^1 \times \mathbb{P}^1$), while for the third (I.9B.5) we specialize to the Clebsch cubic surface. The proofs use the results of Section 6 and Shokurov's connectedness principle. Note that Proposition 7.6 generalizes to the logarithmic setting the result that $\alpha(S) = 2/3$ when $S$ is a cubic surface in

$\mathbb{P}^3$ with an Eckardt point [5]. This result also serves to show (Example 7.7) that the bound of Proposition 6.5 cannot hold without the nefness assumption.

#### 7.2.1 Symmetry considerations

Suppose that $X$ is acted by a finite group $G$ of automorphisms, the divisor $B$ is $G$-invariant, and the class $[H]$ is $G$-invariant. Then one can consider a $G$-invariant analogue of the global lct of the pair $(X, B)$ with respect to $[H]$.

**Definition 7.3.** *Let $G \subset \mathrm{Aut}(X)$. The $G$-invariant global lct of the pair $(X, B)$ with respect to $[H]$ is the number*

$$\alpha_G(X, B, [H]) := \inf\Big\{\mathrm{lct}\big(X, B; D\big) : D \text{ is effective } G\text{-invariant } \mathbb{Q}\text{-divisor such that } D \sim_{\mathbb{Q}} H\Big\},$$

For simplicity, we put $\alpha_G(X, [H]) = \alpha_G(X, B, [H])$ if $B = 0$. Similarly, we put $\alpha_G(X, B_X) = \alpha_G(X, B, [H])$ if $H = -(K_X + B)$. Finally, we put $\alpha_G(X) = \alpha_G(X, [H])$ if $B = 0$ and $H = -K_X$.

#### 7.2.2 $\mathbb{P}^1 \times \mathbb{P}^1$

Let $G$ be a subgroup in $\mathrm{Aut}(\mathbb{P}^1)$ that is isomorphic to $\mathrm{D}_{10}$ (the dihedral group of order 10). Then the action of $G$ is given by an irreducible unimodular two-dimensional representation of the binary dihedral group $2.G$ (a central extension of $G$ by $\mathbb{Z}_2$). Let us denote this representation by $\mathbb{V}_2$ (we can identify it with $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$).

Note that the group $2.G$ has eight distinct irreducible representations: the trivial one (which we denote by $\mathbb{I}$), the two-dimensional representation $\mathbb{V}_2$, three more two-dimensional representations (which we denote by $\mathbb{V}_2'$, $\mathbb{V}_2''$ and $\mathbb{V}_2'''$, and three non-trivial one-dimensional representations (which we denote by $\mathbb{V}_1$, $\mathbb{V}_1'$ and $\mathbb{V}_1''$). Then $\mathrm{Sym}^3(\mathbb{V}_2) \cong \mathbb{V}_2 \oplus \mathbb{V}_2''$. Moreover, one has

$$\mathrm{Sym}^6(\mathbb{V}_2) \cong \mathbb{V}_1 \oplus \mathbb{V}_2' \oplus \mathbb{V}_2' \oplus \mathbb{V}_2''',$$

and $\mathrm{Sym}^2(\mathrm{Sym}^3(\mathbb{V}_2)) \cong \mathrm{Sym}^6(\mathbb{V}_2) \oplus \mathbb{V}_1 \oplus \mathbb{V}_2'''$. This follows from elementary representation theory.

Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^3$ be an embedding given by the linear system $|\mathcal{O}_{\mathbb{P}^1}(3)|$. Then $\phi$ is $G$-equivariant. Put $C = \phi(\mathbb{P}^1)$. Then $C$ is a smooth rational cubic curve in $\mathbb{P}^3$. Since $C$ is projectively normal, we have an exact sequence of $2.G$-representations

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}_C) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_C \otimes \mathcal{O}_{\mathbb{P}^3}(2)) \to 0,$$

where $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \cong \mathrm{Sym}^6(\mathbb{V}_2)$ and $H^0(\mathcal{O}_C \otimes \mathcal{O}_{\mathbb{P}^3}(2)) \cong \mathrm{Sym}^6(\mathbb{V}_2)$. This gives

$$H^0(\mathcal{O}_{\mathbb{P}^3}(2) \otimes \mathcal{I}_C) \cong \mathbb{V}_1 \oplus \mathbb{V}_2''',$$

which implies, in particular, that there exists unique $G$-invariant quadric surface in $\mathbb{P}^3$ that contains the curve $C$. Let us denote this quadric surface by $S$.

Since $C$ is not contained in a hyperplane in $\mathbb{P}^3$, the surface $S$ is reduced and irreducible, Moreover, the surface $S$ is smooth, since $\mathrm{Sym}^3(\mathbb{V}_2)$ does not contain one-dimensional subrepresentations of the group $2.G$. Then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $C$ is a curve of bi-degree $(2, 1)$ on $S$ so $(S, C)$ is (I.4B).

**Proposition 7.4.** *One has $\alpha_G(S, (1 - \beta)C) = 1$ for every $\beta \in (0, 1]$.*

*Proof.* Note that $|-K_S - C|$ is a free pencil on $S$ that gives a projection $S \to \mathbb{P}^1$ (cf. Lemma 4.4). Let $Z_1$ be a curve in $|-K_S - C|$, and let $Z_2, \ldots, Z_r$ be all curves in $|-K_S - C|$ that are images of $Z_1$ via $G$. Then

$$\frac{1}{r}\sum_{i=1}^{r} Z_i + \beta C \sim_{\mathbb{Q}} -(K_S + (1-\beta)C),$$

and $Z_1 + Z_2 + \cdots + Z_r$ is $G$-invariant. On the other hand, we have

$$\mathrm{lct}\big(S, (1-\beta)C; r^{-1}\textstyle\sum_{i=1}^{r} Z_i + \beta C\big) \leq 1,$$

so $\alpha(S, (1-\beta)C) \leq 1$.

Suppose that $\alpha_G(S, (1-\beta)L) < 1$. Then there exists an effective $G$-invariant $\mathbb{Q}$-divisor $\Delta$ such that

$$\Delta \sim_{\mathbb{R}} -(K_S + (1-\beta)C)$$

and the pair $(S, (1-\beta)C + \mu\Delta)$ is not lc at some point $O \in S$ for some positive rational $\mu < 1$. We claim that $(S, (1-\beta)C + \mu\Delta)$ is lc outside of the point $O$. Indeed, suppose that this is not the case. Then $(S, (1-\beta)C + \mu\Delta)$ is not lc along a curve. The latter follows from the connectedness principle [40, Lemma 5.7] since the divisor $-K_S - (1-\beta)C - \mu\Delta)$ is ample, because $\mu < 1$. Thus, we see that there exists a $G$-invariant (possibly reducible) curve $Z \subset S$ such that

$$\Delta = \epsilon Z + \Omega$$

for some effective $\mathbb{R}$-divisors $\Omega$ whose support does not contain the curve $Z$ and some positive rational $\epsilon$ such that either $Z = C$ and $\mu\epsilon > \beta$ or $Z \neq C$ and $\mu\epsilon > 1$. This is, of course, impossible, because $\Delta \sim_{\mathbb{R}} -K_S - C + \beta C$. Indeed, if $Z = C$, then $(\mu\epsilon - \beta)C + \Omega \sim_{\mathbb{R}} -K_S - C$, which implies that

$$0 < 2(\mu\epsilon - \beta) = (\mu\epsilon - \beta)C.(-K_S - C) \leq \Big((\mu\epsilon - \beta)C + \Omega\Big).(-K_S - C) = (-K_S - C)^2 = 0,$$

which is absurd. Thus, we have $Z \neq C$. Then

$$Z.(-K_S - C) \leq \mu\epsilon Z.(-K_S - C) \leq (\mu\epsilon Z + \Omega).(-K_S - C) = (-K_S - C + \beta C).(-K_S - C) = 2\beta,$$

which implies that $Z.(-K_S - C) = 0$. Then $Z \in |n(-K_S - C)|$ for some $n \in \mathbb{N}$. On the other hand, the pencil $|-K_S - C|$ does not contain $G$-invariant curves (if $|-K_S - C|$ contains a $G$-invariant curve, then $|-K_S|$ contains a $G$-invariant curve, which is impossible, since there exists unique $G$-invariant quadric surface in $\mathbb{P}^3$ that contains the curve $C$. Therefore, we see that $n \geq 2$. Then $(n\mu\epsilon - 1)(-K_S - C) + \Omega \sim_{\mathbb{R}} \beta C$, which implies that

$$2 < (2n\mu\epsilon - 1) \leq (2n\mu\epsilon - 1) + \Omega.(-K_S) = ((n\mu\epsilon - 1)(-K_S - C) + \Omega).(-K_S) = \beta C.(-K_S) = 6\beta$$

which is impossible for small $\beta$. The obtained contradiction shows that $(S, (1-\beta)C + \mu\Delta)$ is lc outside of the point $O$.

Since $(1-\beta)C + \mu\Delta$ is $G$-invariant and the pair $(S, (1-\beta)C + \mu\Delta)$ is lc outside of the point $O$, the point $O$ must be $G$-invariant. The latter is impossible, since $\mathrm{Sym}^3(V)$ does not contain one-dimensional sub-representations. Thus $\alpha_G(S, (1-\beta)C) = 1$. ∎

### 7.2.3 $\mathbb{F}_1$

Let $G$ be a subgroup in $\mathrm{Aut}(\mathbb{P}^1)$ that is isomorphic to $\mathrm{D}_{2n}$ (the dihedral group of order $2n$) for $n \geq 2$ (if $n = 2$, then we assume that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Then the action of $G$ is given by an irreducible unimodular two-dimensional representation of the group $2.G$ (a central extension of $G$ by $\mathbb{Z}_2$). Let us denote this representation by $V$. Them $\mathrm{Sym}^2(V)$ is a representation of the group $G$. Moreover, it splits as a union of an irreducible two-dimensional representation of $G$ and a one-dimensional subrepresentation.

Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^2$ be an embedding given by the linear system $|\mathcal{O}_{\mathbb{P}^1}(2)|$. Then $\phi$ is $G$-equivariant. Put $\bar{C} = \phi(\mathbb{P}^1)$. Then $\bar{C}$ is a smooth conic in $\mathbb{P}^2$. Moreover, there exists $G$-invariant point $P \in \mathbb{P}^2$. Since $V$ is irreducible representation of the group $2.G$, we see $P \notin \bar{C}$.

Let $\pi\colon S \to \mathbb{P}^2$ be the blow up of the point $P$. Then the action of $G$ lifts to $S$ and $S \cong \mathbb{F}_1$. Denote by $C$ the proper transform of the curve $\bar{C}$ on the surface $S$. Thus, $(S, C)$ is (I.3A). The proof of the following result is almost identical to the proof of Proposition 7.4.

**Proposition 7.5.** *One has $\alpha_G(S, (1-\beta)C) = 1$ for every $\beta \in (0, 1]$.*

### 7.2.4 Cubic surfaces

The Tian invariant of a smooth cubic surface with an Eckardt point is $2/3$ [5, Theorem 1.7]. The following is a natural generalization.

**Proposition 7.6.** *Let $S$ be a smooth cubic surface in $\mathbb{P}^3$, and let $C$ be a line on $S$. Then the divisor $-(K_S + (1-\beta)C)$ is ample for every real $\beta \in (0, 1]$. Suppose that $C$ contains an Eckardt point. Then*

$$\alpha(S, (1-\beta)C) = \frac{1+\beta}{2+\beta}$$

*for every real $\beta \in (0, 1]$.*

*Proof.* Let $P$ be an Eckardt point on $C$, let $L_1$ and $L_2$ be two lines in $S$ such that $L_1 \cap L_2 \cap C = P$. Then

$$\mathrm{lct}\Big(S, (1-\beta)C; L_1 + L_2 + \beta C\Big) = \frac{1+\beta}{2+\beta}$$

and $L_1 + L_2 + \beta C \sim_{\mathbb{Q}} -(K_S + (1-\beta)C)$, which implies that $\alpha(S, (1-\beta)C) \leq (1+\beta)/(2+\beta)$.

Suppose that $\alpha(S, (1-\beta)C) < (1+\beta)/(2+\beta)$. Then there exists an effective $\mathbb{Q}$-divisor $\Delta$ on the surface $S$ such that $\Delta \sim_{\mathbb{Q}} -(K_S + (1-\beta)C)$ and the pair $(S, (1-\beta)C + \mu\Delta)$ is not lc at some point $O \in S$ for some positive rational number $\mu < (1+\beta)/(2+\beta)$. Let us derive a contradiction (compare the proofs of [5, Lemmas 3.4 and 3.6]).

Since $(1-\beta)C + \Delta \sim_{\mathbb{Q}} -K_S$, it follows from [7, Lemma 5.36] that the pair $(S, (1-\beta)C + \Delta)$ is lc outside of finitely many points in $C$. Hence, the pair $(S, (1-\beta)C + \mu\Delta)$ is lc outside of the a finitely many points in $S$, since $\mu \leq 1$. In fact, this implies that the log pair $(S, (1-\beta)C + \mu\Delta)$ is log canonical outside of the point $O$ by the connectedness principle [40, Lemma 5.7], because the divisor $-(K_S + 1 - \beta)C + \mu\Delta)$ is ample.

If $O \notin C$, then the pair $(S, \mu\Delta)$ is not log canonical at the point $O \in S$, which is impossible, since $\alpha(S) = 2/3$ [5, Theorem 1.7] and $\frac{1+\beta}{2+\beta} < 2/3$. Thus, $O \in C$.

There exists a birational morphism $\pi\colon S \to \mathbb{P}^2$ such that $\pi$ is an isomorphism in a neighborhood of the point $O$, and $\pi(C)$ is a line in $\mathbb{P}^2$. Put $c = \pi(C)$ and $\bar{\Delta} = \pi(\Delta)$. Then the pair $(\mathbb{P}^2, (1-\beta)c + \mu\bar{\Delta})$ is not log canonical at the point $\pi(O)$. Moreover, the pair $(\mathbb{P}^2, (1-\beta)c + \mu\bar{\Delta})$ is lc outside of finitely many points in $\mathbb{P}^2$. Then $(\mathbb{P}^2, (1-\beta)c + \mu\bar{\Delta})$ is lc outside of the point $\pi(O)$ by the connectedness principle, because the divisor $-(K_{\mathbb{P}^2} + (1-\beta)c + \mu\bar{\Delta})$ is ample.

Let $L$ be a general line in $\mathbb{P}^2$. Then the pair $(\mathbb{P}^2, (1-\beta)c + \mu\bar{\Delta} + \epsilon L)$ is not lc along $L$ for every rational number $\epsilon > 1$. Choose $\epsilon > 1$ such that $\epsilon < 1 + 3\beta$. Then

$$3 - (1-\beta) - \mu\big(2+\beta\big) - \epsilon > 3 - 2(1-\beta) - \frac{1+\beta}{2+\beta}\big(2+\beta\big) - \epsilon = 1 + 3\beta - \epsilon > 0,$$

which implies that the divisor $-(K_{\mathbb{P}^2} + (1-\beta)c + \mu\bar{\Delta} + \epsilon L)$ is ample. This contradicts the connectedness principle, because the pair $(\mathbb{P}^2, (1-\beta)c + \mu\bar{\Delta} + \epsilon L)$ is not lc at every point of the non-connected set $\pi(O) \not\in L$, and it is lc outside of this set. $\square$

Proposition 7.6 shows that the nefness conditions in Theorem 6.5 can not be omitted as the following example demonstrates.

**Example 7.7.** Let $S$ be a smooth cubic surface in $\mathbb{P}^3$, and let $C$ be a line on $X$ such that there exists an Eckardt point on $C$. Put $H = -(K_S + (1-\beta)C)$ for any $\beta \in (0,1)$. Then $H$ is ample. Put

$$\gamma = \sup\big\{c \in \mathbb{Q} \bigm| H - cC \text{ is pseudoeffective}\big\}.$$

Then $\gamma = \beta$. Moreover, it follows from Definition 6.3 that $\alpha(S, [H]) \geq \alpha(S, [H + (1-\beta)C]) = \alpha(S) = 2/3$. But it follows from Lemma 6.4 that

$$\alpha(C, [H]|_C)) = \frac{1}{H.C} = \frac{1}{2-\beta}.$$

On the other hand, it follows from Proposition 7.6 that $\alpha(S, (1-\beta)C) = \frac{1+\beta}{2+\beta}$ for any $\beta \in (0,1)$. Thus, we see that

$$\alpha(S, (1-\beta)C, [H]) = \alpha(S, (1-\beta)C) = \frac{1+\beta}{2+\beta} \ngeq \frac{1}{2-\beta} = \min\{\beta/\gamma, \alpha(S,[H]), \alpha(C,[H]|_C)\}$$

for sufficiently small $\beta > 0$. Note that $C$ is not nef, since $C^2 = -1$ on the surface $S$.

Next, we show that for the Clebsch diagonal cubic surface Tian's invariant is in fact equal to 1 for any $\beta \in (0,1]$. Recall that the Clebsch diagonal cubic surface is a smooth cubic surface with $\mathrm{Aut}(S) = \mathrm{S}_5$ (see [22, § 4]). Such surface exists and it is unique (this follows from basic representation and invariant theory of the group $\mathrm{S}_5$).

**Proposition 7.8.** *Let $S$ be the Clebsch diagonal cubic surface, i.e., the unique smooth cubic surface in $\mathbb{P}^3$ such that $\mathrm{Aut}(S) \cong \mathrm{S}_5$. Let $G \cong \mathrm{D}_{10}$ be a subgroup in $\mathrm{Aut}(S)$ consisting of even permutations. Then there exists a $G$-invariant line $C \subset S$ and $\alpha_G(S, (1-\beta)C) = 1$ for every $\beta \in (0,1]$.*

*Proof.* The surface $S$ can be obtained as $\mathrm{A}_5$-equivariant blow up of $\mathbb{P}^2$ at the unique $\mathrm{A}_5$-orbit of length 6 (see [22, § 4] for details). Then the stabilizer in $\mathrm{A}_5$ of any exceptional curve of this blow up is a finite group isomorphic to $G$. Keeping in mind that all finite subgroups in $\mathrm{A}_5$ that are isomorphic to $G$ are conjugate, we see that there exists a $G$-invariant line $C \subset S$.

By Proposition 1.7 the linear system $|-K_S - C|$ is a free pencil of conics on $S$. By our assumptions this pencil is $G$-invariant. Let $Z_1$ be any curve in $|-K_S - C|$, and let $Z_2, \ldots, Z_r$ be all curves in $|-K_S - C|$ that are images of $Z_1$ via $G$. Then the divisor $Z_1 + Z_2 + \cdots + Z_r$ is $G$-invariant and

$$\frac{1}{r}\sum_{i=1}^{r} Z_i + \beta C \sim_{\mathbb{Q}} -K_S - (1-\beta)C.$$

On the other hand,

$$\operatorname{lct}\big(S,(1-\beta)C; r^{-1}\textstyle\sum_{i=1}^r Z_i+\beta C\big)\le 1,$$

so $\alpha(S,(1-\beta)C)\le 1$.

Suppose that $\alpha_G(S,(1-\beta)C)<1$. Then there exists an effective $G$-invariant $\mathbb{Q}$-divisor $\Delta$ such that

$$\Delta\sim_{\mathbb{Q}} -K_S-(1-\beta)C$$

and the pair $(S,(1-\beta)C+\mu\Delta)$ is not lc at some point $O\in S$ for some positive rational $\mu<1$. Since $(1-\beta)C+\Delta\sim_{\mathbb{Q}}-K_S$, it follows from [7, Lemma 5.36] that the pair $(S,(1-\beta)C+\Delta)$ is lc outside of finitely many points in $S$. Since $\mu<1$, the divisor $-K_S-(1-\beta)C-\mu\Delta)$ is ample, and thus the connectedness principle [40, Lemma 5.7] implies that the pair $(S,(1-\beta)C+\mu\Delta)$ is lc outside of the point $O\in S$. In particular, this point $O$ must be $G$-invariant.

On the other hand, the vector space $H^0(\mathcal{O}_S(-K_S))$ is a four-dimensional ($\chi_S(-K_S)=h^0(S,\mathcal{O}_S(-K_S))=1+K_S^2=4$ [18, p. 471] since $S$ is a six-point blow-up of $\mathbb{P}^2$) representation of the group $G$ that splits as a sum of two irreducible two-dimensional representations. Hence, there exists no $G$-invariant point in $S$, since otherwise $H^0(\mathcal{O}_S(-K_S))$ would contain a one-dimensional sub-representation of $G$. Thus $\alpha_G(S,(1-\beta)C)=1$. □

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